Pricing the Default Risk on the Principle of Zero-Utility: Perturbation Method Approximations

by

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Abstract

Tibiletti (2006) uses the zero utility principle to develop insight into the market for default risk. Using general utility functions and a first-order Taylor series approximation to the price of default she develops a shortcut way for the default risk price for both the bargaining counter-parties. If higher order Taylor expansion is carried out, the results are not so simple and the same groups do not appear in both solutions. We suggest a different approach, based on perturbation theory to extend and generalize the results. Initially we assume that the probability of default is known, as Tibiletti did. The results for the seller's price of default involve the same groupings of variables found in the original paper, but higher orders of risk aversion parameters also appear in higher order approximations. Then assumption that the probabilities of insolvency are known by the parties can be relaxed. Since the relations are linear in these probabilities the result is that the probability of insolvency may be replaced by its expected value. Higher moments of the distribution do not enter the determination of the price of default at which the parties are indifferent between undertaking a transaction and foregoing it.

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1. Introduction

Tibiletti (2006) uses the zero utility principle to develop insight into the market for default risk. The expected utility of both buyer and seller are developed under the assumption that the expectation needs to be taken only over the possible outcomes of the transaction. The price of default risk for the two counter-parties is determined to terms of order one in the probability of default and a necessary condition for a fruitful bargaining is then developed.

To extend the results, we investigate two variations of that approach. The first is to use the perturbation method[1] to find higher order approximations, retaining the hypothesis that the probability of default is given. In this context the solution for the price of default of each counter party is assumed to have a Taylor series expansion and the approximations are carried out uniformly in powers of the probability of default. The factors of the Taylor series are obtained by demanding that the coefficient of each power of the probability of default be zero, as must be the case if equality of the two sides is to be preserved. The main advantage in using the perturbation method is that it ensures that both sides of the equation are expanded uniformly in the small parameter, and leads to recursive solutions for the coefficients of successive powers of the small parameter, something that is not achieved by simple Taylor series expansion.

We show that n^{th} order approximation of the risk premium can be expressed by the indices till the n^{th} degree coefficient of risk aversion, where the 2^{nd} degree coefficient is the coefficient of absolute risk aversion, the third is the absolute coefficient of prudence, the fourth is that of temperance, and so on.

At second step we adapt the results to situations in which the expected utility reflects the uncertainty in the estimates of the probability of default as well as the uncertainty in the outcome of the transaction. Thus we deal with the unconditional expected utility, including the sources of uncertainty in real problems, rather than the more limited conditional expectation which assumes that estimated probabilities of insolvency are known with no uncertainty. Because of the linearity of the relations in the probability

[1] See, for example, Bellman, 1966, Cole, 1968.

of default the results are simply to replace the value of these probabilities by its expected value.

The paper is organized as follows. In Section 2 the Tibiletti (2006) results are briefly reviewed. Section 3 presents the perturbation method solutions to both the seller and buyer in the case the probabilities of default are assumed to be known. In Section 4 sensitivity analysis on how uncertainty in the parameters affects the basic equations. Concluding remarks and further research are provided in Section 5.

2. Tibiletti's Findings

In line with the seminal paper of La Valle (1968) and according to classical definitions (Buhlmann, 1970, Gerber, 1979 among others) Tibiletti set out the condition of zero-utility principle from the both counter-parties:

$$u(w_0 - z_s(p)) = (1-p)u(w_0) + pu(w_0 - w) \tag{1}$$

where	$u(w)$	is the seller's utility function,
	w_0	is the seller's initial wealth,
	w	is the loss of wealth in the event of default,
	p	is the seller's estimate of the probability of default, and
	$z_s(p)$	is the seller's price of default.

And that of the buyer as:

$$v(x_0) = (1-q)v(x_0 + z_b(q))qv(x_0 - w + z_b(q)) \tag{2}$$

where	$v(x)$	is the buyer's utility function,
	x_0	is the buyer's initial wealth,
	q	is the buyer's estimate of the probability of default, and
	$z_b(q)$	is the buyer's price of default.

Using Taylor series approximations to order one in the probability of default:

$$z_s(p) = pz'_s\ (0)_s + o(p) \approx z'_s\ (0)p \tag{3}$$

$$z_b(q) = pz_b'(0)_s + o(q) \approx z_b'(0)q \tag{4}$$

She finds that

$$z_s(p) = \frac{u(w_0) - u(w_0 - w)}{u'(w_0 - z_s(p))} p = \frac{u(w_0)}{u'(w_0)} \frac{u(w_0) - u(w_0 - w)}{u(w_0)} p \tag{5}$$

and

$$z_b(p) = \frac{v(x_0) - v(x_0 - w)}{u'(x_0 + z_b(p))} q = \frac{v(x_0)}{v'(x_0)} \frac{v(x_0) - v(x_0 - w)}{v(x_0)} q \tag{6}$$

Both of these are characterized as the product of three factors:

- The "fear of ruin"[2] coefficient at the initial wealth $\left(\frac{u(x_0)}{u'(x_0)} \text{ or } \frac{v(x_0)}{v'(x_0)} \right)$
- The fractional utility variation if default occurs $\left(\frac{u(w_0) - u(w_0 - w)}{u(w_0)} \text{or} \frac{v(x_0) - v(x_0 - w)}{v(x_0)} \right)$, and
- The estimate of the probability of default $(p \text{ or } q)$.

Note that in these derivations the approximation of order $(p \text{ or } q)$ is made by neglecting the price of default. This becomes necessary because the right and left hand sides of the original equations are expanded to different levels of approximation.

[2] This terminology is due to Aumann and Kurz (1977) in their seminal paper on taxation. They refer to the reciprocal of the fear of ruin as "absolute boldness."

3. Solutions Using a Perturbation Method

The problems described above are troubling. A more detailed analysis shows that the uneven expansion is not important in the seller's equation but is important in the buyer's equation, if we take an approximation to the first order in the probability of default as adequate. Perturbation methods provide a way of achieving this and ensure that the solutions are consistent with the assumptions to the level of approximation chosen.

In applying the perturbation method we begin with an equation in which one of the parameters, say λ, is small. Each side of the equation is then expanded into a Taylor series in the small parameter. Assuming that both series converge, it follows from the linear independence of the set λ^i, $i = 1,2,...$ that the coefficients of each power of the small parameter on the two sides must be equal. Using this principle we can recover as many coefficients as we need. As usual the higher the order of derivation, the more cumbersome the calculations. We just carry out the expansions to the cubic term in the small parameter so that the equalities will hold to $o(p^3)$ or $o(q^3)$, nevertheless higher order approximations can obtained.

3.1 The Seller problem

To start the perturbation method with Equation (1) we assume the probability of default p estimated by the seller is small.

Expanding $u(w_0 - z_s(p))$ in a Taylor series in the seller's price of default[3] we have:

$$u(w_0 - z_s(p)) = u(w_0) - u'(w_0) z_s(p) + \frac{1}{2!} u''(w_0)(z_s(p))^2 ... \tag{7}$$

To keep the notation compact we introduce following notation:

$$\begin{aligned} u_0^{(0)} &\equiv u(w_0) \\ u_0^{(k)} &\equiv \left(\frac{\mathrm{d}^k u(t)}{\mathrm{d}t^k}\right)_{t=w_0} \\ \Delta u_0^{(k)} &\equiv \left(\frac{\mathrm{d}^k u(t)}{\mathrm{d}t^k}\right)_{t=w_0} - \left(\frac{\mathrm{d}^k u(t)}{\mathrm{d}t^k}\right)_{t=w_0-w} \end{aligned} \tag{8}$$

[3] This is permissible provided the utility function is continuous in the price of default and its derivatives with respect to this price are also continuous.

Using this notation in Equation (1) we have:

$$\sum_{\mathrm{k}=0}^{\infty}(-1)^k u_0^{(k)}\ z_s{}^k(p) = u_0^{(0)} - p\Delta u_0^{(0)} \tag{9}$$

The solution of the form of a Taylor series[4]:

$$z_s(p) = \sum_{i=0}^{\infty} z_{si}\, p^i \tag{10}$$

Substituting Equation (10) into Equation (9) we get:

$$\begin{aligned} &u_0^{(0)} - u_0^{(1)}\left(z_{s0} + z_{s1}p + z_{s2}p^2 + \ldots\right) + \frac{1}{2!}u_0^{(2)}\left(z_{s0} + z_{s1}p + z_{s2}p^2 + \ldots\right)^2 \\ &+ \frac{1}{3!}u_0^{(3)}\left(z_{s0} + z_{s1}p + z_{s2}p^2 + \ldots\right)^3 \ldots. = u_0^{(0)} - p\Delta u_0^{(0)} \end{aligned} \tag{11}$$

Since the powers of p are linearly independent it follows that the coefficient of every power of p on the right hand side of Equation (11) must equal the coefficients of the corresponding power of p on the left hand side.

We then have the following set of equations to solve to get approximations to the order of p^2:

The coefficient of p^0 in the n^{th} power of $\left(z_{s0} + z_{s1}p + z_{s2}p^2 + \ldots\right)$ must be simply z_{s0}^n, so the equation corresponding to this power will be:

$$\left\{ \begin{aligned} &u(w_0) - u'(w_0)z_{s0} + \frac{1}{2!}u''(w_0)z_{s0}{}^2 + \frac{1}{3!}u'''(w_0)z_{s0}{}^3 \\ &\ldots + \frac{1}{n!}\left(\frac{\mathrm{d}^n u(w)}{\mathrm{d}w^n}\right)_{w=w_0} z_{s0}{}^n + \ldots. \end{aligned} \right\} = u(w_0) \tag{12}$$

But the left hand side is the Taylor series expansion of $u(w_0 - z_{s0})$, so the equation simplifies to $u(w_0 - z_{s0}) = u(w_0)$. This can be true only if $z_{s0} = 0$. This accords with

[4] This is permissible provided the price of risk is continuous in the probability of default and has continuous derivatives with respect to that parameter.

intuition, the perturbation method provides the answer without assumptions.

Given that result, it follows that on the left hand side of Equation (11) the only term of order p^1 arises from the first power of $z_s(p) = (z_{s1}p + z_{s2}p^2 + ...)$, so for this power the equation is:

$$u_0^{(1)} z_{s1} p = p \Delta u_0^{(0)} \tag{13}$$

This can readily be solved to give:

$$z_{s1} = \frac{\Delta u_0^{(0)}}{u_0^{(1)}} = \frac{u(w_0) - u(w_0 - w)}{u'(w_0)} \tag{14}$$

This is the result obtained by Tibiletti.

For the second power of *p*, we obtain the equation:

$$-u_0^{(1)} z_{s2} p^2 + \frac{1}{2!} u_0^{(2)} z_{s1}{}^2 p^2 = 0 \tag{15}$$

So that:

$$z_{s2} = \frac{1}{2!} \frac{u_0^{(2)}}{u_0^{(1)}} z_{s1}{}^2 = -\frac{1}{2} \rho_s(w_0) z_{s1}{}^2 \tag{16}$$

where $\rho_s(w_0)$ is the seller’s coefficient of absolute risk aversion evaluated at w_0.

The third order term is derived analogously and is found to be:

$$\begin{aligned} z_{s3} &= \frac{u_0^{(2)} z_{s1} z_{s2} + \frac{1}{3!} u_0^{(3)} z_{s1}{}^3}{u_0^{(1)}} \\ &= \left(\frac{1}{2!} \frac{u_0^{(2)}}{u_0^{(1)}} - \frac{1}{3!} \frac{u_0^{(3)}}{u_0^{(2)}} \right) \rho_s(w_0) z_{s1}{}^3 \end{aligned} \tag{17}$$

Denoting the seller's k^{th} degree coefficient of risk aversion by $\varphi_s^{(k)}(w_0) = -\frac{u_0^{(k)}}{u_0^{(k-1)}}$, for $k = 2, 3, \ldots$, so the third degree coefficient is the prudence, the third order approximation of the seller's default risk premium may be rewritten:

$$z_s(p) = \frac{u(w_0) - u(w_0 - w)}{u'(w_0)} p - \frac{1}{2}\rho_s(w_0)\left[\frac{u(w_0) - u(w_0 - w)}{u'(w_0)} p\right]^2 \\ -\left[\frac{1}{3!}\varphi_0^{(k)}(w_0) + \frac{1}{2!}\rho_s^2(w_0)\right]\left[\frac{u(w_0) - u(w_0 - w)}{u'(w_0)} p\right]^3 \quad (18)$$

It is worthwhile to note that the each additional term of the approximation includes the n^{th} power of the first-order approximation found by Tibiletti (2006) and depends on three components: (1) the "fear of ruin", (2) the utility variation in percent, and (3) the estimated probability of default. The novelty is that the n^{th} order of approximation includes all coefficients of risk aversion up to the n^{th} degree.

3.2 The Buyer Problem[5]

We expand Equation (2) in a Taylor series in the buyer's price of default. Using a notation analogous to that of Equation (8) we obtain:

$$v_0^{(0)} = \begin{cases} \left[v_0^{(0)} + v_0^{(1)} z_b(q) + \frac{1}{2!} v_0^{(2)} z_b{}^2(q) + \frac{1}{3!} v_0^{(3)} z_b{}^3(q) \ldots\right] \\ -q\left[\Delta v_0^{(0)} + \Delta v_0^{(1)} z_b(q) + \frac{1}{2!}\Delta v_0^{(2)} z_b{}^2(q) + \frac{1}{3!}\Delta v_0^{(3)} z_b{}^3(q) \ldots\right] \end{cases} \quad (19)$$

If the solution is assumed to be of the form:

$$z_b(p) = \sum_{i=0}^{\infty} z_{bi} q^i \quad (20)$$

The coefficient of q^0 yields $v(x_0) = v(x_0 + z_{b0})$, which requires that $z_{b0} = 0$, as we would expect.

[5] This requires the same continuity conditions as the seller case.

Using that result, the coefficient of q^1 yields

$$v_0^{(1)} z_{b1} = \Delta v_0^{(0)} \tag{21}$$

So that

$$z_{b1} = \frac{\Delta v_0^{(0)}}{v_0^{(1)}} = \frac{v(x_0) - v(x_0 - w)}{v'(x_0)} \tag{22}$$

Thus to the first order in the probability parameter the price of default risk of the buyer and seller problems the same form.

The coefficient of q^2 leads to

$$\begin{aligned} z_{b2} &= \frac{1}{2}\left(-\frac{v_0^{(2)}}{v_0^{(1)}}\right) z_{b1}^2 + \frac{\Delta v_0^{(1)}}{v_0^{(1)}} z_{b1} = \left(\frac{1}{2}\rho_b(x_0)\frac{\Delta v_0^{(0)}}{v_0^{(1)}} + \frac{\Delta v_0^{(1)}}{v_0^{(1)}}\right)\frac{\Delta v_0^{(0)}}{v_0^{(1)}} \\ &= \left(\frac{1}{2}\rho_b(w_0)\frac{v(x_0) - v(x_0 - w)}{v'(x_0)} + \frac{v'(x_0) - v'(x_0 - w)}{v'(x_0)}\right)\frac{v(x_0) - v(x_0 - w)}{v'(x_0)} \end{aligned} \tag{23}$$

where $\rho_b(x_0)$ is the buyer's coefficient of absolute risk aversion at x_0.

This has a different form from that of the seller and it is clear that for the buyer the three groups identified by Tibiletti do not always appear together.

The coefficient of q^3 leads to the equation:

$$\begin{aligned} z_{b3} &= \left(-\frac{v_0^{(2)}}{v_0^{(1)}}\right) z_{b1} z_{b2} + \frac{\Delta v_0^{(1)}}{v_0^{(1)}} z_{b2} + \frac{\Delta v_0^{(2)}}{v_0^{(1)}} z_{b1}^2 \\ &= \rho_b(w_0)\left(\frac{1}{2}\rho_b(w_0) + \frac{\Delta v_0^{(1)}}{v_0^{(1)}}\right) z_{b1}^3 + \frac{\Delta v_0^{(1)}}{v_0^{(1)}}\left(\frac{1}{2}\rho_b(w_0) + \frac{\Delta v_0^{(1)}}{v_0^{(1)}}\right) z_{b1}^2 + \frac{\Delta v_0^{(2)}}{v_0^{(1)}} z_{b1}^2 \end{aligned} \tag{24}$$

Once again, the three Tibiletti groups do not appear together. The risk aversion coefficient and the coefficient of prudence enter the relation with the second and third order of the estimated probability of default.

4. Accounting for Uncertainty in the Probability Estimates

The development given above is purely academic in the sense that it assumes the two actors make estimates but have no uncertainty about the estimates. Estimates, however, are uncertain by definition. To reflect this fact we need to obtain expectations not only over the outcome of the transaction but over the uncertainty in the parameters. The problem requires careful specification of what probability is being assessed and for what purpose.

Two points of view could be sustained. One, to which we will refer as "subjective assessment" is that each counter-party aims to estimate **its own** probability of default and the zero utility cost of default corresponding to that probability. The second one, to which we will refer as "objective assessment", is that each counter-party aims to assess **the market's** assessment of the probability of default and the zero-utility corresponding to market's assessment.

4.1 "Subjective Assessment"

If the seller's subjective estimate of the probability of default is uncertain, and the outcome is independent of the error of estimation, then the seller's equation should be modified to reflect this. The conditional expectation theorem[6] provides a useful tool in this context. It leads directly to the relation:

$$
\begin{aligned}
u\left(w_0 - {}_S z_s\right) &= \int \left[(1-p)\; u(w_0) + p u(w_0 - w)\right] \mathrm{d}F(p) \\
&= (1 - \mathrm{E}[p])\; u(w_0) + \mathrm{E}[p] u(w_0 - w)
\end{aligned}
\qquad (25)
$$

where the prefix E emphasizes that price of risk reflects the fact that effect of errors in the subjective estimate of the probability of insolvency is included. In this case we re-define the right hand side to reflect the expected utility given the uncertainty in the estimate. Note that there is no need to express the price of risk as a function of the uncertain probability of insolvency. Doing so would imply that the price of risk is set by the actual probability of insolvency rather than by the seller's perception of it.

The fact that the right hand side is linear in p results in a very simple relation, the function ${}_S z_s$ is a function only of the expected value of p and can be obtained from

[6] See, for example, Ross, 2007.

the series for $z_s(p)$ by replacing the probability by its expected value.

Similarly, the buyer's equation would become:

$$v(x_0) = \int_{q=0}^{q=1} \left[(1-q)v(x_0 + {}_S z_b) + qv(x_0 - w + {}_E z_b)\right] \, dG(q)$$
$$= (1 - \mathrm{E}\{q\})v(x_0 + {}_S z_b) + \mathrm{E}[q]v(x_0 - w + {}_E z_b) \qquad (26)$$

Once again, the linearity implies that the solution is obtained by replacing the value of q by $\mathrm{E}[q]$ in the solution for the deterministic case.

4.1 "Objective Assessment"

In this case we need to consider the effect of uncertainty in the market's valuation of the probability on the price of risk so we need to modify Equation 25 to

$$\int_{p=0}^{p=1} u(w_0 - {}_O z_s(p))dF(p) = (1 - E[p]) \; u(w_0) + E[p]u(w_0 - w) \qquad (27)$$

Where the prefix O emphasizes that the uncertainty in the objective assessment in the estimate is included.

Expanding this we obtain:

$$\int_{p=0}^{p=1} \left[u(w_0) - u_0^{(1)} \, {}_O z_s(p) + \frac{1}{2!} u_0^{(2)} \, {}_O z_s^2(p) - \ldots \right] \, dF(p)$$
$$= (1 - E[p]) \; u(w_0) + E[p] \; u(w_0 - w) \qquad (28)$$

$$\int_{p=0}^{p=1} \left[u(w_0) - u_0^{(1)} \left({}_O z_{s0} + {}_O z_{s1} p + \ldots \right) + \frac{1}{2!} u_0^{(2)} \left({}_O z_{s0} + {}_O z_{s1} p + + \ldots \right)^2 - \ldots \right] \, dF(p)$$
$$= (1 - E[p]) \; u(w_0) + E[p]u(w_0 - w) \qquad (29)$$

The terms independent of p on the left hand side equal $u(w_0 - {}_O z_{s0})$ and those on the right they equal $u(w_0)$ which again leads to the conclusion that ${}_O z_s(0) = {}_O z_{s0} = 0$. For the higher order terms we are left with:

$$\int_{p=0}^{p=1}\left[\begin{array}{l}-u_0^{(1)}\left({}_O z_{s1}p+{}_O z_{s2}p^2+\ldots\right)+\dfrac{1}{2!}u_0^{(2)}\left({}_O z_{s1}p+{}_O z_{s2}p^2+\ldots\right)^2\\ -.\dfrac{1}{3!}u_0^{(3)}\left({}_O z_{s1}p+{}_O z_{s2}p^2+\ldots\right)^3+\dfrac{1}{4!}u_0^{(4)}\left({}_O z_{s1}p+{}_O z_{s2}p^2+\ldots\right)^4..\end{array}\right]dF(p)$$

$$= -E[p](u(w_0)-u(w_0-w)) \tag{30}$$

$$\left[\begin{array}{l}-u_0^{(1)}\left({}_O z_{s1}E[p]+{}_O z_{s2}E[p^2]+{}_O z_{s3}E[p^3]+{}_O z_{s4}E[p^4]+\ldots\right)\\ +\dfrac{1}{2!}u_0^{(2)}\left({}_O z_{s1}^2E[p^2]+2{}_O z_{s1}\,{}_O z_{s2}E[p^3]+\left({}_O z_{s2}^2+2{}_O z_{s1}\,{}_O z_{s3}\right)E[p^4]+\ldots\right)\\ -.\dfrac{1}{3!}u_0^{(3)}\left({}_O z_{s1}^3E[p^3]+3{}_O z_{s1}^2\,{}_O z_{s2}E[p^4]+\ldots\right)+\ldots\end{array}\right]$$

$$= -E[p](u(w_0)-u(w_0-w)) \tag{31}$$

If we can use the same rationale as in the deterministic case and equate the coefficients of the expected value of corresponding powers of p, then we have:

$${}_O z_{s1} = \frac{(u(w_0)-u(w_0-w))}{u'(w_0)} = z_{s1} \tag{32}$$

which leads to the same first order term as the deterministic case.

For the second power we get

$$-u_0^{(1)}\,{}_O z_{s2} + \frac{1}{2!}u_0^{(2)}\,{}_O z_{s1}{}^2 = 0 \tag{33}$$

which leads to the result:

$${}_O z_{s2} = \frac{1}{2!}\frac{u_0^{(2)}}{u_0^{(1)}}z_{s1}{}^2 = -\frac{1}{2!}\rho_s(w_0)z_{s1}{}^2 = z_{s2} \tag{34}$$

And the third power leads to

$$-u_0^{(1)}\,{}_O z_{s3} + u_0^{(2)}z_{s1}z_{s2} -.\frac{1}{3!}u_0^{(3)}z_{s1}^3 = 0 \tag{35}$$

which gives:

$$
{}_O z_{s3} = \frac{u_0^{(2)} z_{s1} z_{s2} + \frac{1}{3!} u_0^{(3)} z_{s1}{}^3}{u_0^{(1)}}
$$

$$
= \left(\frac{1}{2!} \rho(w_0) - \frac{1}{3!} \frac{u_0^{(3)}}{u_0^{(2)}} \right) \rho(w_0) z_{s1}{}^3 = z_{s3} \tag{36}
$$

While the coefficients have the same form that they did before, the form of the cost function is quite different:

$$
{}_O z_s(p) = \frac{u(w_0) - u(w_0 - w)}{u'(w_0)} E[p] - \frac{1}{2} \rho_s(w_0) \left[\frac{u(w_0) - u(w_0 - w)}{u'(w_0)} \right]^2 E[p^2]
$$

$$
- \left[\frac{1}{3!} \varphi_s(w_0) + \frac{1}{2!} \rho_s^2(w_0) \right] \left[\frac{u(w_0) - u(w_0 - w)}{u'(w_0)} \right]^3 E[p^3] + \ldots
$$

$$
= \frac{u(w_0) - u(w_0 - w)}{u'(w_0)} \mu_1(p) - \frac{1}{2} \rho_s(w_0) \left[\frac{u(w_0) - u(w_0 - w)}{u'(w_0)} \right]^2 \left(\mu_2(p) + \mu_1^2(p) \right)
$$

$$
- \left[\frac{1}{3!} \varphi_s(w_0) + \frac{1}{2!} \rho_s^2(w_0) \right] \left[\frac{u(w_0) - u(w_0 - w)}{u'(w_0)} \right]^3 \left(\mu_3(p) + 3\mu_1(p)\mu_2(p) + \mu_1^3(p) \right) \tag{37}
$$

$$
+ \ldots
$$

$$
= z_{s1}\mu_1(p) + z_{s2}\mu_1^2(p) + z_{s3}\mu_1^3(p) + \ldots + z_{s2}\mu_2(p) + 3z_{s3}\mu_1(p)\mu_2(p) + \ldots
$$

$$
+ z_{s3}\mu_3(p) + \ldots
$$

Thus we may need approximations well beyond the third to get an accurate representation, especially so if the expected value is small but higher moments are not small. It is also clear that if the variance is of the order of the square of the expected value the first order term in the variance is of the same order of magnitude as the term in the second order of the expected value.

Similarly, the buyer's Equation 26 would become:

$$
v(x_0) = \int_{q=0}^{q=1} \left[(1-q) v\left(x_0 + {}_O z_b(q)\right) + q v\left(x_0 - w + z_b(q)\right) \right] \, dG(q) \tag{38}
$$

When expanded, this becomes:

$$v_0^{(0)} = \left\{ \begin{array}{l} \int\limits_{q=0}^{1} \left[v_0^{(0)} + v_0^{(1)}\,{}_O z_b(q) + \frac{1}{2!} v_0^{(2)}\,{}_O z_b{}^2(q) + \frac{1}{3!} v_0^{(3)}\,{}_O z_b{}^3(q) \ldots \right] dG(q) \\ - \int\limits_{q=0}^{1} q \left[\Delta v_0^{(0)} + \Delta v_0^{(1)}\,{}_O z_b(q) + \frac{1}{2!} \Delta v_0^{(2)}\,{}_O z_b{}^2(q) + \frac{1}{3!} \Delta v_0^{(3)}\,{}_O z_b{}^3(q) \ldots \right] dG(q) \end{array} \right\} \tag{39}$$

And expanding ${}_O z_b(q)$ into its Taylor series we find:

$$v_0^{(0)} = \left\{ \begin{array}{l} \int\limits_{q=0}^{1} \left[\begin{array}{l} v_0^{(0)} + v_0^{(1)} \left({}_O z_{b0} + {}_O z_{b1} q + {}_O z_{b2} q^2 + \ldots \right) + \frac{1}{2!} v_0^{(2)} \left({}_O z_{b0} + {}_O z_{b1} q + {}_O z_{b2} q^2 + \ldots \right)^2 \\ + \frac{1}{3!} v_0^{(3)} \left({}_O z_{b0} + {}_O z_{b1} q + {}_O z_{b2} q^2 + \ldots \right)^3 \ldots \end{array} \right] dG(q) \\ - \int\limits_{q=0}^{1} q \left[\begin{array}{l} \Delta v_0^{(0)} + \Delta v_0^{(1)} \left({}_O z_{b0} + {}_O z_{b1} q + {}_O z_{b2} q^2 + \ldots \right) + \frac{1}{2!} \Delta v_0^{(2)} \left({}_O z_{b0} + {}_O z_{b1} q + {}_O z_{b2} q^2 + \ldots \right)^2 \\ + \frac{1}{3!} \Delta v_0^{(3)} \left({}_O z_{b0} + {}_O z_{b1} q + {}_O z_{b2} q3 + \ldots \right)^2 \ldots \end{array} \right] dG(q) \end{array} \right\} \tag{40}$$

This again leads to exactly the same relations as the deterministic case, but with the expected value of the k^{th} power of the probability rather than the k^{th} power probability itself. As in the case of the seller, the series reduces to an infinite series in powers of the expected value plus infinite series in powers of the variance, third central moment and so on. More than three terms may be required to get good approximations especially if the distribution of the estimate is antimodal, that is, is highly concentrated in the neighborhoods of zero and one, since in that case the higher central moments are of the same order of magnitude as the expected.

5. Discussion

The simple perturbation method used in this paper leads readily to the estimation of the polynomial coefficients in the uniform expansion of the price of default risk for the seller and the buyer.

When the probability of default is assumed to be given, the perturbation solution for the seller's price of default risk preserves, at least to terms of order three, the structure that Tibiletti (2006) found, in the sense that her three factors appear together. In general, the n^{th} order term of the default price can be expressed as a function of risk aversion coefficients up to the n^{th} degree. Under the same conditions the buyer's price of default does not preserve the same three groups beyond the first term; the coefficients of risk aversion, prudence, temperance, and so on enter is somewhat more complicated ways.

The fact that the relations between utility before the transaction and utility after the transaction are both linear in the probability of default leads to a simple solution in the event that the probability represents an estimate rather than a certain value. In both cases we obtain the same series development but is uncertainty us involved we need to use the expected value of the probability of default, as seen from the point of view of the decision maker, in the place of the deterministic value. The higher moments of the estimate do not play a role in the price at which bargaining would be fruitful.

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