

# Optimal Reinsurance Arrangements Under Tail Risk Measures

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## Abstract

Regulatory authorities demand insurance companies to control the risks by imposing stringent risk management policies. This article investigates the insurance company's optimal risk management strategy subject to regulator's risk measure constraints. We first design the optimal reinsurance contracts under different tail risk measures. Then we analyze the impact of the regulators' requirements on the way insurers and reinsurers share risks. Our results underline some adverse incentives when requirements are based on the Value-at-Risk or the Conditional Tail Expectation risk measure. Our findings confirm recent empirical studies (for instance Froot (2001)) which show that insurers do not often purchase coverage for high layers of risk. Alternative risk measure might be more appropriate to request insurance companies to concentrate on large amount of risk. Finally we provide alternative risk transfer mechanisms on the capital market.

*Keywords:* Optimal Reinsurance, Risk Measures, Alternative Risk Transfer

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**Abstract:** Regulatory authorities demand insurance companies to control the risks by imposing stringent risk management policies. This article investigates the insurance company's optimal risk management strategy subject to regulator's risk measure constraints. We first design the optimal reinsurance contracts under different tail risk measures. Then we analyze the impact of the regulators' requirements on the way insurers and reinsurers share risks. Our results underline some adverse incentives when requirements are based on the Value-at-Risk or the Conditional Tail Expectation risk measure. Our findings confirm recent empirical studies (for instance Froot (2001)) which show that insurers do not often purchase coverage for high layers of risk. Alternative risk measure might be more appropriate to request insurance companies to concentrate on large amount of risk. Finally we provide alternative risk transfer mechanisms on the capital market.

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# Introduction

Risk averse insureds purchase insurance policies to increase the expected utility of their final wealth. Arrow (1963, 1971), Borch (1971), Mossin (1968), Raviv (1979) lay out the fundamental principles of the design of the optimal insurance policies in a general maximum expected utility framework. This expected utility model has been extended extensively by many authors such as Aase (2002), Gollier (1987, 1996), Schlesinger (1997), Doherty and Schlesinger (1983, 1985), and has been successfully used to explain the incentive to insure and design the optimal insurance policies for risk averse insured.

However, the expected utility framework could not illustrate the insurance demand for risk neutral firms (see Eeckhoudt, Gollier and Schlesinger (2005)). Indeed, at first sight, risk neutral firms have no desire to buy costly insurance since expected utility of the final wealth would be decreased by purchasing an insurance contract and firms could eliminate insurable risk through diversification. In this paper we will examine the prevalence of the reinsurance demand and derive the optimal risk management strategies for risk neutral insurance companies.

There have been many empirical studies to explain why risk neutral corporations purchase insurance even though their conclusions are more or less controversial. Mayers and Smith (1982) found out that companies want to avoid bankruptcy costs or have tax incentives. They also discovered that insurance demand can reduce regulatory constraints on firms. Yamori (1999) empirically observed that Japanese corporations can have a low default probability and a high demand for insurance. Chen and PonArul (1989) showed that tax incentives may be not so significant as expected through simulations. According to Davidson, Cross and Thornton (1992), the first reason for the corporate purchase of insurance lies in the bondholder's priority rule. On the other hand, Hoyt and Khang (2000) argue that corporate insurance purchases are driven by agency conflicts, tax incentives, bankruptcy costs and regulatory constraints. Recently, Hau (2006) provides a simple framework to explain how liquidity can explain the property insurance demand. See, for instance H. Seog (2006), and G. Plantin (2006) for more discussions.

In this paper we undertake a different view on the reinsurance demand by focusing on the impact of risk management policies. Regulators introduce several risk management methodologies (such as Solvency II) to protect policyholders and monitor closely how insurance companies implement those risk management programs. For this purpose, regulators often impose some risk measures and the regulated insurance companies must satisfy those risk measure constraints along their business lines. We will derive the optimal reinsurance coverage subject to those risk measure constraints. It turns out that with the presence of the risk management constraint imposed by the regulators, risk neutral insurance companies behave risk averse and those insurance companies have desire to share their risks with a reinsurer in the reinsurance market or the capital market. This

fact has been recognized by Caillaud, Dionne and Jullien (2000) in another context, in which they derive the optimal financial contract to invest in a risky project<sup>1</sup>.

Our contribution in this paper is to derive the optimal design of reinsurance contracts given some risk measure constraints stemming from regulators. There are three major findings in this paper. *First*, we present the optimal risk management strategy for insurance companies by incorporating both controlled risk and objective to maximize profit. We design the optimal contracts that satisfy risk management constraints while maximize expected wealth, or equivalently, minimize the reinsurance premium. *Second*, we discuss the impact of regulators' choice on the optimal coverage of the reinsurance contracts and assess regulators' influence on the risk management decisions taken by companies. We show how adverse incentives can come from tail risk measures such as Value-at-Risk (VaR) or Conditional Tail Expectation (CTE). Our findings confirm recent empirical studies (for instance Froot (2001)) which show that insurers do not often purchase coverage for high layers of risk. *Third*, we show how a stronger control of regulators leads to the optimality of deductible. We also compare our results with contracts observed in both insurance market and the capital market.

The results in this paper have implications in other contexts. In the presented study, both insurance companies and reinsurance companies are supposed to be risk neutral. A non-expected utility framework is developed by Doherty and Eeckhoudt (1995), Gollier and Schlesinger (1996), Schlesinger (1997). Since those previous works are based on Yaari's theory and have a different viewpoint apart from us, our approach forms a complement to the literature on the non-expected utility framework. On the other hand, in a subsequent article we are able to solve the optimal coverage design for both risk averse insurer and risk averse reinsurer under risk management measures, by extending the method of this paper.

The organization of this paper is as follows. In the next section, a theoretical framework is introduced, and we will derive the optimal reinsurance contract under VaR risk measure. Discussions of our theoretical result will be presented in the section heading with "Discussions". In this section we first investigate the risk/reward profile of the optimal insurance contract, then present some numerical examples to illustrate the impact on insurance demand of risk management constraints. In the "Optimal Reinsurance Arrangements under CTE and other Risk Measures" section we will solve the optimal reinsurance design problem under other risk measures. We show that, surprisingly, the optimal coverage under CTE measure is similar to the optimal coverage under VaR measure. However, under an alternative risk measure, a deductible contract is optimal. In "Reinsurance Market, Capital Market " section we compare the optimal reinsurance contracts under risk measures to contracts frequently sold by reinsurers in the market. Then we discuss alternative risk transfer to the capital market. Finally we extend the analysis to a ran-

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<sup>1</sup>Full coverage above a straight deductible, in their framework, is interpreted as *induced risk aversion*.

dom maturity situation. The “Conclusions” section concludes, and the “Appendix” section provides the proofs.

# 1 Optimal Reinsurance Design under VaR Measure

Throughout the paper, the “insured” refers to the insurance company while the word “insurer” is used to refer to the reinsurer. Insured is motivated to buy a reinsurance contract from insurer to meet risk measure limits imposed by regulators. A reinsurance contract is characterized by the premium paid by the insured and by a coverage function (or indemnity) specifying the transfer from the insurer to the insured for each possible loss. In this section, we first describe the framework, then derive the optimal coverage of reinsurance arrangement under VaR risk measure. Discussions under other risk measures will be presented in the “Optimal Reinsurance Arrangements under CTE and other Risk Measures” section.

## 1.1 Framework

The model is single-period. At the beginning of the period, an insurance company receives premia (total amount equal to  $\Pi$ ) from its customers, and in exchange to the premia, it has to provide its customers a coverage at the end of the period. The aggregate amount of indemnities paid at the end of the period is denoted by  $X$ . If  $C$  is the initial own capital of the company, its initial wealth is  $W_0 = C + \Pi$ . Its final wealth, at the end of the period, is

$$\widehat{W} = W_0 - X \tag{1}$$

The insurance company faces a risk of large loss. It is requested by regulators to meet risk management requirements to protect the interests of both policyholders and shareholders.<sup>2</sup> As an example, assume that  $v$  is a VaR limit to confidence level  $\alpha$  imposed by regulators, then the VaR constraint for the insurance company is  $P(W_0 - \widehat{W} > v) \leq \alpha$ . Equivalently,  $P(X > v) \leq \alpha$ . Obviously this VaR requirement is violated if  $P(X > v) > \alpha$  without reinsurance market.

We now assume that the insurance company purchase a reinsurance contract from a reinsurer, paying an initial premium  $\mu$ . When  $X$  is observed, an indemnity  $I(X)$  is transferred from reinsurer to insurer. Let us denote by  $W$  the insurance company’s final wealth after entering the reinsurance market. Then we have

$$W = W_0 - \mu - X + I(X) \tag{2}$$

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<sup>2</sup>We assume no default risk in this paper. On the other hand, rating agencies currently not only focus on the default and volatility of the insurance company, but also look into the risk management of the regulated insurance company. Therefore the presence of the rating agencies also introduce the risk aversion of insurer by our presented analysis. We thank Harry Panjer for providing us this argument.

The indemnity  $I(X)$  is understood as a function of the loss variable  $X$ . Following classical insurance literature (see for example Arrow (1971), Doherty and Eeckhoudt (1995), Gollier (2003) and Raviv (1979)), the coverage  $I(X)$  is nonnegative and can not exceed the size of the loss.

In the design of the reinsurance agreement, we assume that the premium is based on the actuarial value of the policy plus a proportional loading. Precisely, the premium  $\mu$  is determined by:

$$\mu = (1 + \rho)E[I(X)] \quad (3)$$

where  $\rho > 0$  is called the loading factor. Some previous authors don't assume any form of the premium principle. Technically speaking, our subsequent results are true as well without the premium principle.<sup>3</sup> The presented analysis could be also extended to more general premium principle  $\mu = g(E[I(X)])$  where  $g$  is increasing, non negative and satisfies  $g(x) \geq x$  and  $g(0) = 0$ .

The final loss  $L$  of the insurance company is:

$$L = W_0 - W = \mu + X - I(X) \quad (4)$$

$L$  represents the sum of the premium  $\mu$  initially paid and the retention of the loss. The VaR constraint in this case is formulated as  $P(L > v) \leq \alpha$ . It equivalent to  $VaR_L(\alpha) \leq v$ <sup>4</sup>.

Some questions arise naturally at this stage. Is it correct that the higher premium paid, the smaller the probability  $P(L > v)$ ? If  $P(X > v) \geq \alpha$ , is this possible to purchase a reinsurance contract such that the VaR constraint is satisfied? What is the optimal premium and optimal reinsurance contract for the insured to meet the VaR constraint? The purpose of this section is to examine these questions.

A simple reflection implies that the answer to the first question is not always true. For instance, if the premium  $\mu$  is strictly greater than the VaR limit  $v$ , then the loss  $L = \mu + X - I(X) > v$  because of the property of the indemnity  $I(X)$ . Thus  $P(L > v) = 1 > \alpha$ ! Hence, the VaR condition doesn't satisfy at all if the insurance company pays a premium which is greater than the VaR limit. The available premium  $\mu$  of purchasing an insurance contract subject to VaR constraint is thus bounded by the VaR limit  $v$  (for the discussion of the relationship between premium and the probability we refer to Remark 1.2 below).

We now assume that  $P(X > v) \geq \alpha$  and examine the problem of purchasing a reinsurance contract to meet VaR constraint. Inspiring by Arrow's classical work (Arrow (1963,

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<sup>3</sup>In fact, while the premium is not linked to the derivations of these results are simpler and could be treated as the first step in our Two-Step solving process. However, Raviv (1979) explained why the premium received is required by regulation to be a function of the policy's actuarial value  $E[I(X)]$ .

<sup>4</sup>VaR is defined by  $VaR_L(\alpha) = \inf\{x, P(L > x) \leq \alpha\}$ .

1971)), we consider first the deductible contract with coverage  $I_d(X) = (X - d)^+$  for some non negative deductible  $d$ . If the insurance company purchases a deductible contract as above, then the total wealth of the insurance company is

$$W^d = W_0 - X - \mu + I_d(X) = W_0 - \mu - \min\{X, d\} \quad (5)$$

Thus the VaR constraint becomes  $P(\min\{X, d\} > v - \mu) \leq \alpha$ . In some special cases, deductible contract is one solution for the insurance company. For example, if the actuarial value of  $X$  is small enough such that  $(1 + \rho)E[X] \leq v$ , then we can choose  $I_d(X) = X$  to satisfy the VaR constraint. In general, among the deductible contracts  $I_d(X)$ , we are lead to discuss the following static minimization problem

$$\min_{d \geq 0} \{ P(\min\{X, d\} > v - (1 + \rho)E[I_d(X)]) \} \quad (6)$$

However, as will be proved in Prop 1.1 shortly, the deductible contract is not optimal to purchase under VaR constraint. Thus we need to explore methods different from Arrow (1971).

To proceed, we consider the following optimization problems:

$$(A) \quad \min_{\beta \geq 0, I(X)} \{ P(W < W_0 - v) \} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] \leq \beta \end{cases} \quad (7)$$

and

$$(A_\beta) \quad \min_{I(X)} \{ P(W < W_0 - v) \} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] \leq \beta \end{cases} \quad (8)$$

where the problem  $(A_\beta)$  solves the minimal probability of the event that loss  $L$  is strictly larger than the VaR limit by paying premium at most  $(1 + \rho)\beta$  to purchase a reinsurance contract, and problem  $(A)$  solves the minimal probability of the event that loss  $L > v$  among all possible insurance contracts by ignoring the premium issue. Clearly, if the solution of problem  $(A)$  is strictly greater than  $\alpha$ , it is impossible for the insurance company to meet the VaR constraint. Problem  $(A)$  is reduced to consider problem  $(A_\beta)$  for all possible  $\beta$ .

There is a nice interpretation of the problem  $(A_\beta)$  and its dual problem for risk neutral insured. Note that

$$E[W] = W_0 - E[X] - \mu + E[I(X)] = W_0 - E[X] - \rho E[I(X)] \quad (9)$$

Therefore

$$E[I(X)] \leq \beta \iff E[W] \geq W_0 - E[X] - \rho\beta \quad (10)$$

Hence the problem  $(A_\beta)$  also solves the minimal probability of  $\{L > v\}$  subject to a minimal guaranteed expected return of the final wealth. The problem  $(A_\beta)$  thus represents the trade-off between the expected wealth return and the probability  $P(L > v)$ .

Consequently, its dual problem  $(A^\alpha)$  can be interpreted as follows:

$$(A^\alpha) \quad \max_I (E[W]) \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ P(L > v) \leq \alpha \end{cases} \quad (11)$$

The dual problem  $(A^\alpha)$  maximizes the expected final wealth, or equivalently minimizes the premium, subject to VaR constraint. In terms of the indemnity  $I(X)$  only,  $(A^\alpha)$  can be reformulated as follows:

$$(A^\alpha) \quad \min_I (E[I(X)]) \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ P(I(X) > X - v + (1 + \rho)E[I(X)]) \geq 1 - \alpha \end{cases} \quad (12)$$

The problem  $(A^\alpha)$  is intuitively appealing because it addresses the minimal possible premium while the VaR constraint is considered. However, the last expression in (12) of this problem is not standard because the objective function  $E[I(X)]$  is involved inside the constraint conditions. Moreover, the solution of  $(A^\alpha)$  doesn't lead to a "robust" construction of the optimal indemnity. The reason is as follows. Given one indemnity  $I(X)$  subject to the VaR constraint and  $E[I(X)] = \beta$ , we can add small non-negative adjustment of  $I(X)$  over the "bad" set  $\mathcal{A} := \{I(X) > X - v + (1 + \rho)\beta\}$ , and reduce the indemnity over its complementary set  $\mathcal{A}^c$  such that the total expectation is the same as  $\beta$ . Therefore one can construct somewhat arbitrarily another indemnity with the same premium to satisfy the same VaR constraint. Later we will construct one optimal indemnity  $I(X)$  naturally by using the solution of the problem  $(A_\beta)$ .

We will focus on the problem  $(A_\beta)$  in the subsequent discussion.

## 1.2 Optimal Reinsurance Under VaR

The problem  $(A_\beta)$  is solved in two steps. First, the premium is assumed fixed and the optimal reinsurance coverage is derived in terms of the premium. Second, the optimal premium and thus the optimal reinsurance coverage are determined by solving a static minimization problem. The method has been used by Raviv (1979) and others to determine the optimal design when the premium principle is either assumed or not assumed. Precisely, we first discuss the problem

$$(\bar{A}_\beta) \quad \min_{I(X)} \{ P(W < W_0 - v) \} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] = \beta \end{cases} \quad (13)$$

Then the problem  $(A_\beta)$  is reduced to solve the problem  $(\bar{A}_{\beta_1})$  for all possible  $\beta_1 \leq \beta$ . Proposition 1.1 below presents the optimal coverage  $I$  of the optimization problem  $(A_\beta)$  and problem  $(\bar{A}_\beta)$ .

**Proposition 1.1** *Assuming  $X$  is a non negative real random variable that has a continuous cumulative distribution function*<sup>5</sup>. *Let*

$$\mathcal{C} = \{ \beta : 0 \leq \beta < E[(X - v + (1 + \rho)\beta)^+], (1 + \rho)\beta < v \} \quad (14)$$

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<sup>5</sup>We allow the case when there is a mass point at 0 meaning  $P(X = 0)$  can be positive.

1 . For any  $\beta \in \mathcal{C}$ , there exists a positive  $\lambda_\beta^{var}$  such that the optimal coverage  $I_\beta^{var}(X)$  solves the optimization problem  $(\bar{A}_\beta)$ , where

$$I_\beta^{var}(X) = \begin{cases} 0 & \text{if } X < v - (1 + \rho)\beta \\ X + (1 + \rho)\beta - v & \text{if } v - (1 + \rho)\beta \leq X \leq v - (1 + \rho)\beta + \frac{1}{\lambda_\beta^{var}} \\ 0 & \text{if } X > v - (1 + \rho)\beta + \frac{1}{\lambda_\beta^{var}} \end{cases}$$

$\lambda_\beta^{var}$  is determined such that:

$$E [I_\beta^{var}(X)] = \beta. \quad (15)$$

2 . Define

$$\mathcal{P}(\beta) = P \left( X > v - (1 + \rho)\beta + \frac{1}{\lambda_\beta^{var}} \right), \quad \forall \beta \in \mathcal{C} \quad (16)$$

Then  $I_{\beta_1}^{var}, \beta_1 \leq \beta$  solves the problem  $(A_\beta)$  where  $\beta_1$  solves the static minimization problem

$$\min_{0 \leq \beta_1 \leq \beta} \mathcal{P}(\beta_1) \quad (17)$$

According to Prop 1.1, the optimal reinsurance coverage subject to the VaR constraint involves a deductible for small loss, and coinsurance (actually no insurance) for large loss. We call it “truncated stop loss”. A detailed analysis of this result will be presented in the next section. To finish the discussion of this section, some remarks are given in order.

**Remark 1.1** *A Simple Description of  $\mathcal{C}$ .*

Since the set  $\mathcal{C}$  plays a key role in Proposition 1.1 and subsequent discussions in Proposition 3.1 and 3.2, we first present a simple description of this set  $\mathcal{C}$ . Clearly  $0 \in \mathcal{C}$  and  $\mathcal{C} - \{0\}$  is an open set. It is easy to see that

$$\{\beta : 0 \leq \beta < E[(X - v + \beta)^+], (1 + \rho)\beta < v\} \subseteq \mathcal{C} \quad (18)$$

and the subset  $\{\beta : 0 \leq \beta < E[(X - v + \beta)^+], (1 + \rho)\beta < v\}$  is of the form  $[0, a_\beta)$  for some positive number  $a_\beta$ . In general, the set  $\mathcal{C}$  might be not necessary an interval. Hence the second part of Prop 1.1 should be understood for *relatively small*  $\beta$ .<sup>6</sup> If  $\beta$  doesn't belong to  $\mathcal{C}$ , it is not obvious yet regard to the existence of the optimal solution of the problem  $(\bar{A}_\beta)$  as well as  $(A_\beta)$ .

**Remark 1.2**

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<sup>6</sup>Same remarks apply to both Prop 3.1 and Prop 3.2 below.

Note that

$$\mathcal{P}(\beta) = P(W_\beta^{var} < W_0 - v)$$

where  $W_\beta^{var} = W_0 - X - (1 + \rho)\beta + I_\beta^{var}(X)$ .  $\mathcal{P}(\beta)$  represents the minimal probability of  $\{L > v\}$  by purchasing insurance contracts via exactly premium amount  $(1 + \rho)\beta$  for  $\beta \in \mathcal{C}$ . It is tempting to think that the higher premium, the smaller probability  $\mathcal{P}(\beta)$ , and then the optimal premium in the problem  $(A_\beta)$  is  $(1 + \rho)\beta$ . In other words, the premium constraint is binding in the problem  $(A_\beta)$ . The following simple example illustrates that it is not true in general. To see this point, consider a random variable  $X$  uniformly distributed over  $[0, 1]$ . To simplify the notation we use  $t$  to denote the parameter  $\frac{1}{\lambda^{var}}$ . Hence  $t$  is determined by the equation

$$\beta = E[(X + (1 + \rho)\beta - v)^+ \mathbb{1}_{X \leq v - (1 + \rho)\beta + t}] \quad (19)$$

Let  $\beta \in \mathcal{C}$ . According to the assumption on the loss variable  $X$  (which density function is  $\mathbb{1}_{[0,1]}(x)$ ), it is then easy to see that  $\beta$  satisfies  $\beta \leq \frac{v}{1 + \rho}$  and  $v - (1 + \rho)\beta + \sqrt{2\beta} < 1$  and that  $t$  is equal to:

$$t = \sqrt{2\beta} \quad (20)$$

In the present situation, we have

$$\begin{aligned} \mathcal{P}(\beta) &= P(X > v - (1 + \rho)\beta + t) \\ &= 1 - v + (1 + \rho)\beta - \sqrt{2\beta} \end{aligned}$$

For  $\beta \in \mathcal{C}$ , clearly  $\mathcal{P}(\beta)$  is decreasing when  $\beta \leq \frac{1}{(1 + \rho)^2}$ , and then increasing when  $\beta \geq \frac{1}{(1 + \rho)^2}$ . Therefore,  $\min_{0 \leq \beta_1 \leq \beta} \mathcal{P}(\beta_1)$  is not always achieved at  $\beta$ . For example if  $v = 0.65$ ,  $\rho = 0.6$  then  $\mathcal{C} = \left(0, \frac{v}{1 + \rho}\right)$  (because the condition  $v - (1 + \rho)\beta + \sqrt{2\beta} < 1$  is always true). Since  $\frac{v}{1 + \rho} \approx 0.406$ , the minimum is obtained at  $\frac{1}{(1 + \rho)^2} \approx 0.391$ .

**Remark 1.3** *Solvency Interpretation*

In particular  $v = W_0$ , the problem  $(A_\beta)$  could be interpreted in terms of ruin probability:

$$\min_I P(W < 0) \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] \leq \beta \end{cases}$$

The objective is to minimize the probability of ruin at maturity of the issuing company. Proposition 1.1 states that the company gives up to insure itself above a given level, which corresponds to the default probability. After ruin, insurance is useless. In this special case this problem was considered by Gajek and Zagrodny (2004b). Even though they discovered the same optimal shape, their construction of the optimal coverage was not explicit since they used a different method.

## 2 Discussions

In this section we first illustrate Proposition 1.1 in the context of expected wealth return. The maximum expected wealth return subject to a VaR constraint is not observed in previous literature, and the optimal coverage follows easily from Proposition 1.1. Then we will address several business issues regard to the VaR management. We will illustrate Proposition 1.1 using a numerical example. At last we explain the impact of the regulator on the reinsurance market.

### 2.1 Risk & Reward

We now assume the insurance company wants to construct a reinsurance contract such that the VaR limit is just satisfied, say  $P(W < W_0 - v) = \alpha$ .<sup>7</sup> By Prop 1.1, the optimal contract with premium  $(1 + \rho)\beta$  is  $I_\beta^{var}(X)$ . Moreover, because  $P(W < W_0 - v) = \alpha$ , we have

$$v - (1 + \rho)\beta + \frac{1}{\lambda_\beta^{var}} = q \quad (21)$$

where  $P(X \leq q) = 1 - \alpha$ ,  $q$  is the  $(1 - \alpha)$  quantile of the distribution of  $X$ . Therefore, the optimal coverage  $I_\beta^{var}(X)$  could be written as

$$I^\alpha(X) = \begin{cases} 0 & \text{if } X < v - (1 + \rho)\beta \\ X + (1 + \rho)\beta - v & \text{if } v - (1 + \rho)\beta \leq X \leq q \\ 0 & \text{if } X > q \end{cases} \quad (22)$$

The premium  $\mu = (1 + \rho)\beta$  is determined by the premium principle:

$$E[(X + (1 + \rho)\beta - v)^+ \mathbf{1}_{X \leq q}] = \beta \quad (23)$$

and the minimal premium for the insured corresponding to the minimal solution of the above equation (23).

Assume the premium  $\mu$  is determined, the optimal contract  $I^\alpha(X)$  is determined by the deductible level  $d = \mu - v$ . The loss  $L = W_0 - W$  is given by the formula:

$$L = \begin{cases} \mu + X & \text{if } X < d \\ \mu + d & \text{if } d \leq X \leq q \\ \mu + X & \text{if } X > q \end{cases} \quad (24)$$

Figure 1 shows the indemnity  $I^\alpha(X)$ . Figure 2 illustrates the total loss  $L$  with arbitrary values for the premium ( $\mu = 2$ ), the VaR limit ( $v = 5$ ), the deductible ( $d = v - \mu = 3$ ) and the  $(1 - \alpha)$ -quantile ( $q = 7$ ).

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<sup>7</sup>Since the VaR constraint is  $P(W < W_0 - v) \leq \alpha$ , then the consideration of the condition  $P(W < W_0 - v) \leq \alpha$  leads insurance company takes advantage of the VaR risk management role.

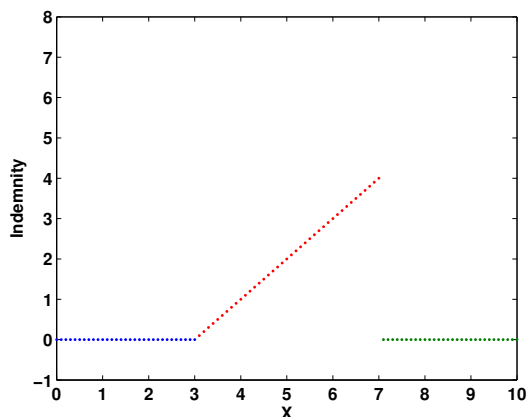


Figure 1: Indemnity  $I(X)$  w.r.t.  $X$ .  
Parameters are set to  $v = 5$ ,  $\mu=2$ ,  $q = 7$ .

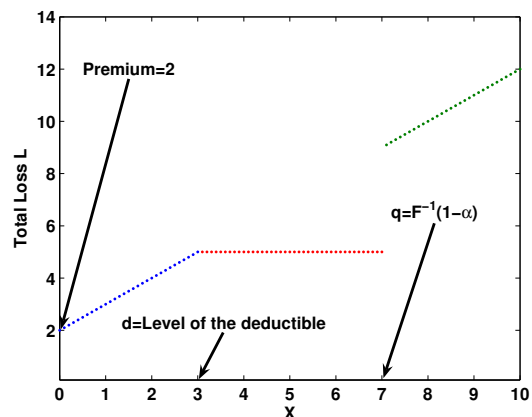


Figure 2: Total Loss  $L$  w.r.t.  $X$ .  
Parameters are set to  $v = 5$ ,  $\mu=2$ ,  $q = 7$ .

To summarize, the insured needs to pay premium  $(1 + \rho)\beta$  by purchasing the contract  $I^\alpha(X)$  and the VaR condition is just binding.  $I^\alpha(X)$  is *optimal* for insured in the sense that other indemnities  $I(X)$  with the same premium would violate the VaR constraint.<sup>8</sup>

### Remark 2.1

The above discussion is consistent with Arrow's results when the insured is risk neutral. Indeed, the criteria in Arrow's framework is to maximize the expected utility of the final wealth. It is obtained when the risk constraint disappears (in (11)), that is  $\alpha \rightarrow 1$ , then  $q \rightarrow 0$ . Hence no insurance is optimal because of insurance costs.

## 2.2 Several Business Issues

We now discuss several business features of the optimal coverage under VaR constraint.

### 2.2.1 VaR Risk Management

Adopting an optimal reinsurance arrangement under VaR constraint, insurance companies choose to leave the worst states uninsured (see Figures 1 and 2). This result is reasonable giving the maximum expected wealth objective, since those worst states are in fact the most expensive ones to insure against. Actually, such contracts reduce the probability to incur large loss but they do not limit the losses' amount in worst states. Thus VaR requirements lead companies to ignore losses in the tail of the distribution. A deeper analyze of the consequences of Value-at-Risk management can be found in Basak and Shapiro (2001). They prove that VaR risk managers often choose larger risk exposure

<sup>8</sup>Because the VaR constraint is binding for  $I^\alpha(X)$ , then by Prop 1.1, any other indemnity with the same premium leads to  $P(W < W_0 - v) > \alpha$ .

to risky assets and consequently incur larger losses when losses occur. Regulators would however prefer that most adverse states of the world are considered and that managers try to limit their exposure towards large losses. To address this issue, they propose the alternative model of risk management based on the optimization of some conditional tail expectation. We will discuss the optimal insurance contract under other risk management model in the next section.

### 2.2.2 Moral Hazard

In case the insured can partly control the losses' amount, the optimal design obtained in proposition 1.1 is subject to moral hazard. This is evident to see by observing the payoff function of the coverage  $I^\alpha(X)$  or  $I_\beta^{var}(X)$ . In purchasing the contract with coverage  $I^\alpha(X)$ , insureds would never declare a loss more than  $q$  to the insurance company because they receive nothing if they follow the contract arrangement. To avoid moral hazard, the contract could be modified and designed as follows:

$$I^*(X) = \begin{cases} 0 & \text{if } X < v - \mu \\ X + \mu - v & \text{if } v - \mu \leq X \leq q \\ q + \mu - v & \text{if } X > q \end{cases}$$

This contract is a well known contract involving a stop loss rule with an upper limit on coverage.<sup>9</sup> Therefore the actual insurance contract comprised both the optimal design and moral hazard. We will illustrate the difference between the two contracts with same premium  $\mu$  through a numerical example in subsection 2.3 below. In subsection 2.3, Figure 3 illustrates the difference between the optimal contract and the modified contract. Figure 4 displays the final wealth in both cases.

### 2.3 An Example

Let us illustrate the optimal contract through a numerical example. We assume  $X$  is a LogNormal random variable ( $\mathcal{LN}(m, \sigma^2)$ ), that is  $X = e^Z$  where  $Z$  is a gaussian random variable ( $\mathcal{N}(m, \sigma^2)$ ). Let  $\Phi$  be the cumulative distribution function of a standard gaussian variable  $\mathcal{N}(0, 1)$ , then  $\Phi_{m, \sigma^2}(x) = \Phi\left(\frac{x-m}{\sigma}\right)$  is the cumulative distribution function of  $Z$ .

$$P(X \leq x) \geq 1 - \alpha \quad \Leftrightarrow \quad x \geq q \triangleq \exp\left(\Phi_{m, \sigma^2}^{-1}(1 - \alpha)\right)$$

$X$  includes the costs and the reimbursements to the insured. For the sake of simplicity, we suppose the pricing is fair, the total amount  $\Pi$  received from policyholders at the beginning of the period is equal to:

$$\Pi = E[X] = e^{m + \frac{\sigma^2}{2}} \approx 60,174$$

---

<sup>9</sup>Policies with upper limit on coverage could be derived from minimizing some risk measures under a mean variance premium principle. See for example Gajek and Zagrodny (2004a).

m	$\sigma$	$W_0$	$\rho$	$\alpha$
10.4	1.1	100,000	0.15	5%

Table 1: Parameters

Then shareholders invest  $W_0 - \Pi \approx 39,826$ . Assume:

$$v = \frac{3}{2}\Pi \approx 90,261 \quad ; \quad W_0 - v \approx 9,739$$

Let  $\alpha$  be the confidence level decided by regulators,  $\alpha = 5\%$ . Then the VaR constraint  $P(W < W_0 - v) \leq \alpha$  is imposed by regulators.

### 2.3.1 In the absence of reinsurance

After collecting its premia, the company has an initial wealth  $W_0$  subject to a possible loss  $X$ . If the insurance company does not purchase insurance, then its final wealth is  $\widehat{W} = W_0 - X$ . The company's expected final wealth and the shareholders' expected return  $\eta$  is:

$$E[\widehat{W}] \approx 39,826 \quad \eta = \frac{E[\widehat{W}]}{W_0 - \Pi} = 1. \quad (25)$$

Given the level  $v = 90,261$ , the probability a loss greater than  $v$  occurs is equal to 17.92%. Hence the above regulator imposed VaR constraint doesn't satisfy. The insurance company need to find a solution to satisfy regulators' requirements. We examine in the next paragraph the consequences of purchasing reinsurance. Note that the return is 1 since the pricing is fair. In fact, we neglect the margin profit that is added to the premium and shared between shareholders.

### 2.3.2 With reinsurance

The insurance company's final wealth is now equal to  $W = W_0 - \mu - X + I(X)$ . Since  $\alpha = 5\%$ , the  $(1 - \alpha)$ -quantile of the LogNormal distribution of  $X$  is equal to  $q \approx 200,650$ . By previous discussion, the optimal coverage is equal to:

$$I^\alpha(X) = (X - v + \mu)\mathbb{1}_{X \in [v-\mu, q]}$$

Due to assumptions on the premium principle, we are looking for a premium  $\mu$  satisfying:

$$\frac{\mu}{1 + \rho} = \int_{v-\mu}^q (x - v + \mu) dP_X(x)$$

Elementary computations<sup>10</sup> show that condition  $(1 + \rho) \int_0^q x dP_X(x) < v < q$  is a sufficient condition to have existence of a solution  $\mu \in (0, v)$ . We solve numerically this equation

<sup>10</sup>It suffices to study the continuous mapping  $\mu \rightarrow \int_{v-\mu}^q (x - v + \mu) dP_X(x)$ . It is increasing on  $(0, v)$ . Note that it is positive at  $\mu = 0$  and less than  $\frac{v}{1+\rho}$  at  $\mu = v$  if  $(1 + \rho) \int_0^q x dP_X(x) < v$ .

and find  $\mu \approx 7,026$ , and the expected final wealth is equal to

$$E[W] \approx 38,909 \quad \eta = \frac{E[W]}{W_0 - \Pi} \approx 0.97 \quad (26)$$

Comparisons between results (25) and (26) show obvious consequences of purchasing reinsurance (a lower expected wealth and a lower expected return for the shareholders).

### 2.3.3 Comparison of two contracts

Under the same range of parameters, we compare the optimal contract and the deductible with an upper limit. We assume the premium  $\mu$  is the same and equal to  $\mu \approx 7,026$ .

On Figure 3, the plain line corresponds to the optimal contract under a VaR constraint and the dash line is the capped contract. They have the same premium  $\mu$ . The upper limit is equal to 56,212 (maximum reimbursement from reinsurer). The deductible is computed by  $d = v - \mu = 83,235$ .

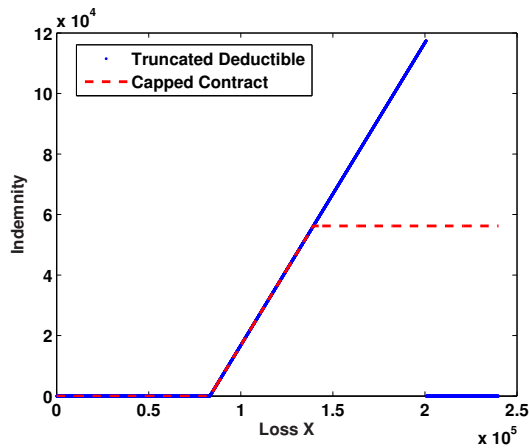


Figure 3: Indemnity  $I(X)$  w.r.t.  $X$

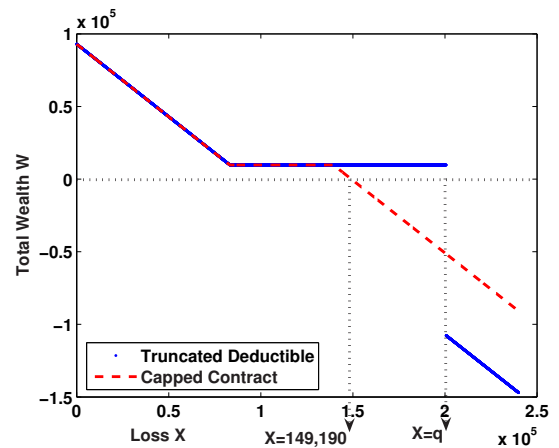


Figure 4: Final Wealth  $W$  w.r.t.  $X$

Figure 4 displays the final wealth in both cases. Clearly, truncated deductible and capped stop loss lead to unbounded loss. When the observed loss amount  $X$  tends to infinity, the final wealth tends to  $-\infty$ . We clearly note that the optimal contract protects the company against bankruptcy up to an observed loss equal to  $q = 200,650$  whilst the deductible with an upper-limit protects the company up to a loss amount of 149,190. Insurance is useless after ruin. If the company cannot control the amount of loss, the truncated deductible is then possible and seems clearly to be a better design.

## 2.4 Trade-Off Risk & Return, Insurer-Regulator Interaction

### 2.4.1 Induced Risk Aversion.

Shareholders want to maximize the expected return of the insurance company but have to satisfy regulators' requirements. Let us represent in Figure 5 and 6 these two opposite objectives.

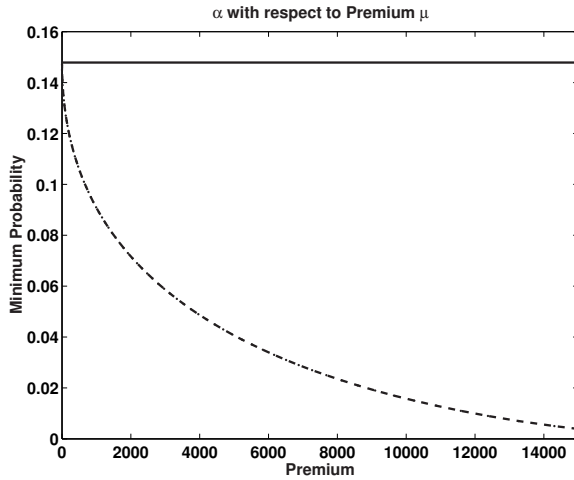


Figure 5: Probability  $\alpha$  w.r.t.  $\mu$

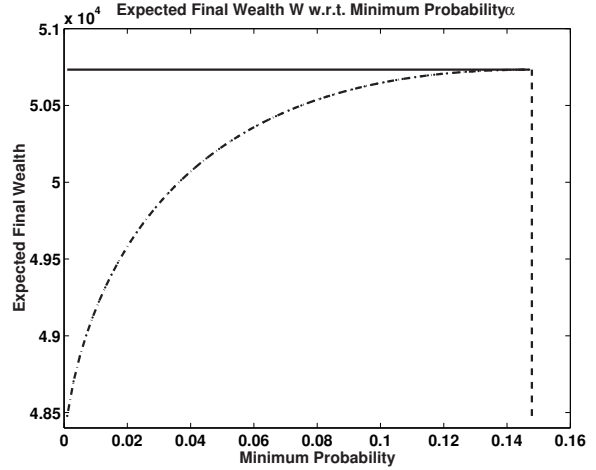


Figure 6: Expected Final Wealth w.r.t. Risk

In Figure 5, we show the impact of the regulators' choice ( $\alpha$ ) on the insurance demand. The greater the insurance demand (premium  $\mu$ ), the higher the coverage and the less the probability to be below the reference level at the end of the period. The horizontal line represents this probability in case no insurance is purchased. In Figure 6, we link the expected return (through the final expected wealth) with the risk (measured by the level  $\alpha$ ). The observed shape is concave similar to the trade-off between risk and return for a risk-averse investor. Because of the presence of regulatory constraints, risk neutral insurance companies react as a risk-averse individual (*induced risk aversion* is explained by Caillaud, Dionne and Jullien (2000) in another context). Maximum risk and maximum expected final wealth is obtained when no insurance is purchased.

### 2.4.2 Initial Capital $W_0$ .

In order to decrease their reinsurance demand, insurance companies can also increase their initial capital  $W_0$ . This feature is illustrated by Figure 7 and Figure 8.

Figure 7 illustrates also the trade-off between risk and return (as figure 5) but for different initial wealth  $W_0$ . The higher  $W_0$ , the higher is the expected return since less reinsurance has to be purchased.

Figure 8 provides answer to the following question: given a ruin probability (say fixed by regulation authorities), how much does the company initially need in order to guarantee

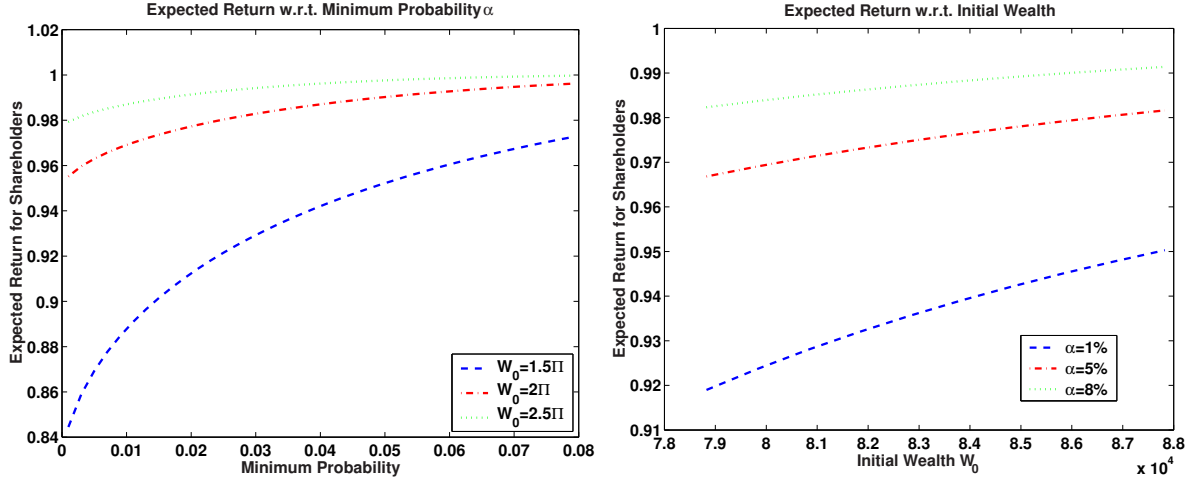


Figure 7: Expected Return w.r.t. Probability Figure 8: Expected Return with respect to  $P(W < W_0 - v)$  for different initial capital  $W_0$  Initial Wealth  $W_0$

an expected wealth greater than a minimum guarantee  $G$ ? Given  $\alpha$  one can easily find the initial wealth  $W_0$  that will give the willing expected return.

The initial capital is  $W_0$  and we solve the issue of the minimum initial capital, given a minimum expected guarantee  $G$  and a maximum probability of ruin  $\alpha$ .

Note that this analysis does not take into account the cost of the economic capital and assume premia paid by policyholders are fair. If there is a profit margin, say for example 5%, then the profit 5% of  $\Pi$  has to be shared between shareholders. The more they are, the less they get! Then it could be more interesting to buy reinsurance than to increase the initial capital.

### 2.4.3 Regulators' Impact

From regulators' viewpoint, two parameters are important. First the minimum capital requirement  $W_0 - v$  and second, the probability  $\alpha$  to be below this minimum at the end of the period (see Figure 9 and 10). The VaR constraint writes as:

$$P(W < W_0 - v) \leq \alpha$$

Consequences of regulators' choice are illustrated by graphs 9 and 10. Indeed they first influence *the demand for insurance*. That is, how much does the company spend to purchase reinsurance? We measure the reinsurance demand through the percentage of the collected premia used to buy reinsurance arrangements.

$$\frac{\mu}{\Pi}$$

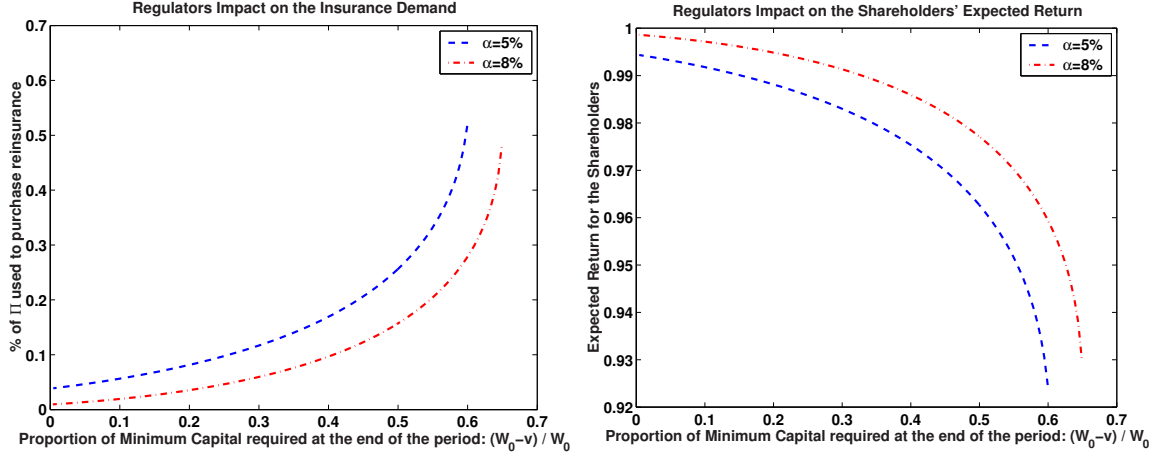


Figure 9: Insurance Demand w.r.t. to Figure 10: Expected Return w.r.t. to Regulators' Requirements: Levels  $\alpha$  and Regulators' Requirements: Levels  $\alpha$  and  $W_0 - v$ .

Second they also control *shareholders' expected return* that is equal to:

$$\frac{E[W]}{W_0 - \Pi}.$$

On the one hand, Figure 9 shows that the demand for reinsurance increases with the minimum capital required at the end of the period and with the confidence level  $(1 - \alpha)$  that losses will not exceed the pre-specified level. On the other hand, Figure 10 illustrates the fact that the more severe the requirements are, the lower the expected return is. Both graphs show that regulators' constraints play a strategic role in the company's risk management decisions.

### 3 Optimal Reinsurance Arrangements under CTE and other Risk Measures

We have explored the optimal reinsurance contract under VaR constraint. In this section we will derive the optimal reinsurance contract under other risk measures. We first discuss the CTE risk measure.

#### 3.1 CTE Risk Measure

According to Section 1, the optimal insurance doesn't insure large loss if the insurance company meets VaR constraint. It is well established that CTE risk measure has many theoretical advantages (see Artzner etc (1999), Basak and Shapiro (2001)) as well as practical usefulness (To quote only a few, Boyle, Hardy and Vorst (2005) enlighten interesting

properties of the conditional tail expectation compared to the Value at Risk. See also Inui and Kijima (2005)). We wonder whether the CTE risk measure imposed by regulators have different impact on the insurance company and the reinsurance market. This is the purpose of this subsection.

The optimal design under the conditional tail expectation can be stated as follows.

$$(B_\beta) \quad \min_I \{E[(W_0 - W)\mathbf{1}_{W_0 - W > v}]\} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] \leq \beta \end{cases} \quad (27)$$

where  $\mu = (1 + \rho)\beta$  is determined, and the insurance company investigates the minimal possible expected loss by paying premium at most  $\mu = (1 + \rho)\beta$ . Similar to the previous discussions for VaR risk measure, the solution of problem  $(B_\beta)$  also involves two steps. We first solve the problem  $(B_\beta)$  by fixing the premium, then reduce the problem  $(B_\beta)$  to a static minimization problem of real number variable. Put

$$(\bar{B}_\beta) \quad \min_I \{E[(W_0 - W)\mathbf{1}_{W_0 - W > v}]\} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] = \beta \end{cases} \quad (28)$$

**Proposition 3.1** *Assuming  $X$  has a continuous cumulative distribution function strictly increasing on  $[0, +\infty)$ .*

1 . Assume that  $\beta \in \mathcal{C}$ , then there exists a positive  $\lambda_\beta^{cte}$  such that the coverage  $I_\beta^{cte}$  solves optimization program  $(\bar{B}_\beta)$ .

$$I_\beta^{cte}(X) = \begin{cases} 0 & \text{if } X(\omega) < v - (1 + \rho)\beta \\ X(\omega) + (1 + \rho)\beta - v & \text{if } v - (1 + \rho)\beta \leq X(\omega) \leq v - (1 + \rho)\beta + \frac{v}{\lambda_\beta^{cte} - 1} \\ 0 & \text{if } X(\omega) > v - (1 + \rho)\beta + \frac{v}{\lambda_\beta^{cte} - 1} \end{cases}$$

$\lambda_\beta^{cte} > 1$  and it is determined such that:

$$E[I_\beta^{cte}(X)] = \beta.$$

2 . Define

$$\begin{aligned} \mathcal{T}(\beta) &:= E\left[(W_0 - W_\beta^{cte})\mathbf{1}_{W_0 - W_\beta^{cte} > v}\right] \\ &= E\left[\{(1 + \rho)\beta + X\}\mathbf{1}_{X(\omega) > v - (1 + \rho)\beta + \frac{v}{\lambda_\beta^{cte} - 1}}\right] \end{aligned}$$

where the final wealth  $W_\beta^{cte}$  is derived from  $I_\beta^{cte}(X)$ . Then  $I_{\beta_1}^{cte}(X)$  solves the problem  $(B_\beta)$  where  $\beta_1 \in [0, \beta]$  solves the static optimization problem

$$\min_{0 \leq \beta_1 \leq \beta} \mathcal{T}(\beta_1) \quad (29)$$

As is well known, CTE risk measure has many advantages over the VaR risk measures (see Basak and Shapiro (2001)). Hence Proposition 3.1 is surprising since it derives similar optimal coverage as in Proposition 1.1 under VaR constraint. Proposition 3.1 shows that the insurance companies have no incentive to protect themselves against large losses under the conditional tail expectation's constraint. Both Proposition 1.1 and 3.1 provide theoretical foundation to Froot (2001)'s empirical investigation. But from the perspective of regulator, constraints on the expected shortfall are not enough to incite companies to purchase insurance against catastrophic risks. We will back to this issue shortly in the next subsection.

**Remark 3.1** *Level of the deductible*

The optimal contracts  $I_{\beta}^{var}(X)$  and  $I_{\beta}^{cte}(X)$  are “truncated deductible”. The level of the “truncated deductible” is easy to be interpreted. Let  $d = \mu - v$  in both cases (VaR and CTE). The total loss  $L$  is the sum of the premium  $\mu$  and the retention of the loss (that is  $X - I(X)$ ). We want to have a probability less than  $\alpha$  to have a loss more than  $v$ . It turns out to look at a retention less than  $d = v - \mu$  since  $\mu$  is a certain loss.

$$I^*(X) = \begin{cases} 0 & \text{if } X < d \\ X - d & \text{if } d \leq X \leq q \\ 0 & \text{if } X > q \end{cases}$$

### 3.2 Emphasize the Right Tail Distribution

We have shown that despite its interesting properties, the CTE risk measure gives us similar results and leads to the same adverse behaviors as the Value-at-Risk constraint. It is interesting to investigate whether fully deductible contract is optimal under some risk measure. In this subsection we introduce a risk measure which is based on the expected square of the excessive loss.

Precisely, we examine the expected square of the excessive loss.

$$(C_{\beta}) \quad \min_I \left\{ E \left[ (W_0 - W - v)^2 \mathbf{1}_{W_0 - W > v} \right] \right\} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] \leq \beta \end{cases} \quad (30)$$

Similar to the VaR measure, this expected square of excessive loss measure is not a coherent risk measure in the sense of Artzner et al. (1999). But this measure does pay attention on the loss amount. After a simple modification of the CTE measure we show that regulators have to place more weight on the right tail of the loss distribution in order to incite companies to purchase insurance against large losses.<sup>11</sup>

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<sup>11</sup> Alternatively, one would use some distortion function to increase the impact of the right tail of the distribution (see Wirch and Hardy (1999)) but the corresponding optimization problem seems hard to solve.

Proposition 3.2 below solves the Problem  $(C_\beta)$ . Let

$$(\bar{C}_\beta) \quad \min_I \left\{ E \left[ (W_0 - W - v)^2 \mathbf{1}_{W_0 - W > v} \right] \right\} \quad \text{s.t.} \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] = \beta \end{cases} \quad (31)$$

**Proposition 3.2** *Assuming  $X$  has a continuous cumulative distribution function strictly increasing on  $[0, +\infty)$ .*

1 . *Assume that  $\beta \in \mathcal{C}$ . Then there exists a positive  $\lambda_\beta^*$  such that the coverage  $I_\beta^*$  solves optimization program  $(\bar{C}_\beta)$ .*

$$I_\beta^*(X) = \begin{cases} 0 & \text{if } X(\omega) < v - (1 + \rho)\beta + \frac{\lambda_\beta^*}{2} \\ X(\omega) + (1 + \rho)\beta - v - \frac{\lambda_\beta^*}{2} & \text{if } v - (1 + \rho)\beta + \frac{\lambda_\beta^*}{2} \leq X(\omega) \end{cases}$$

where  $\lambda_\beta^* > 0$  is determined by:

$$E[I_\beta^*(X)] = \beta.$$

2 . *Define*

$$\begin{aligned} \mathcal{S}(\beta) &:= E \left[ (W_0 - W_\beta^* - v)^2 \mathbf{1}_{W_0 - W_\beta^* > v} \right] \\ &= E \left[ ((1 + \rho)\beta + X - v)^2 \mathbf{1}_{\left\{ v - (1 + \rho)\beta < X < v - (1 + \rho)\beta + \frac{\lambda_\beta^*}{2} \right\}} \right] \\ &\quad + \left( \frac{\lambda_\beta^*}{2} \right)^2 P \left( X \geq v - (1 + \rho)\beta + \frac{\lambda_\beta^*}{2} \right) \end{aligned}$$

where the final  $W_\beta^*$  is derived from  $I_\beta^*(X)$ . Then  $I_{\beta_1}^*(X)$  solves the problem  $(C_\beta)$  where  $\beta_1 \in [0, \beta]$  solves the static optimization problem

$$\min_{0 \leq \beta_1 \leq \beta} \mathcal{S}(\beta_1) \quad (32)$$

Note that the total loss is bounded contrary to the truncated stop loss we obtained in the previous subsections. The optimal design is a deductible, it seems to be more reasonable and not subject to the same moral hazard. However, the following numerical example shows this design could be very costly and is maybe not the best solution.

### 3.3 An Example: Continued

In this subsection we compare the stop loss and the truncated stop loss with a numerical example. We know that stop loss is optimal under the expected square of excess loss. We use data given in subsection 2.3 (see table 1). With  $v$  and  $\alpha$  fixed, we showed before that the insurance company is protected against a loss up to 95%-quantile  $q = 200,650$  (which is more than the total initial wealth  $W_0$ ). Thanks to the reinsurance arrangement

(with the truncated stop loss) the insurance company avoids bankruptcy and has “only” the deductible amount to pay ( $v - \mu = 90,261 - 7,026 = 83,235$ ) which is already a huge amount compared to  $W_0 = 100,000$ .

Assume that the insurance company decides to buy a stop loss contract with the same premium  $\mu = 7,026$ , then the level  $d^*$  of the deductible has to satisfy

$$\mu = (1 + \rho) \int_{d^*}^{+\infty} (x - d^*) dP_X(x).$$

Numerically, we solve this equation and obtain  $d^* \approx 235,220$ ! The level of the deductible should be less than  $W_0$ , otherwise the insurer is ruined before the reinsurance arrangement starts working!

If the company wants to be fully protected, it has to find the maximum deductible level  $d$  such that the company is not ruined when this level is triggered (indeed if  $d = W_0$  then the company has to pay  $\mu(d) = (1 + \rho)E[(X - d)^+]$  at the beginning such that its wealth becomes  $W_0 - \mu(d)$ , it is not enough to provide the deductible amount  $W_0$ . Numerically, we solve  $d + \mu(d) = W_0$ . We find  $d = 79,703$ . When  $d = 79,703$ , then the premium  $\mu(d) = 20,297$ . It represents more than 30% of the collected premia  $\Pi$ . Full protection is costly.

## 4 Reinsurance Market, Capital Market

In this section we first compare our results to traditional reinsurance policies, then interpret the reinsurance arrangements as a portfolio of derivatives written on a loss index and finally extend our results to design contracts with a random maturity.

### 4.1 Comparing Existing Reinsurance Contracts and our Results

Froot (2001) underlines that most reinsurance arrangements are “excess-of-loss layer” with a retention level (the deductible, level that losses must exceed before coverage is triggered), a limit (the maximum amount reimbursed by the reinsurer) and an exceeding probability (probability losses are above the limit). The contract writes as:

$$I_1(X) = (X - d)\mathbb{1}_{X \in [d, l]} + (l - d)\mathbb{1}_{X > l}.$$

where  $d$  is the deductible level,  $l$  stands for the upper limit of the coverage, thus  $l - d$  is the maximum indemnity. Actually, we show that to avoid moral hazard (see figures 3 and 4) this design could be a solution. Moral hazard can also be reduced by a coinsurance treaty, where the coverage is partly provided by the reinsurer (for instance  $\theta = 95\%$ ) (see Cummins, Lalonde and Phillips (2004))

$$I_\theta(X) = \theta(X - d)\mathbb{1}_{X \in [d, l]} + \theta(l - d)\mathbb{1}_{X > l}, \quad \theta < 1.$$

The recent work of Froot (2001) gives also an overview of the market for catastrophe risk. He notices that “most insurers purchase relatively little cat reinsurance against large events and that premiums are high relative to expected losses”. He explains that both reinsurance and CAT bonds generally trade at significant margins above the expected loss and that insurers tend to retain rather than share their large event risks. This is in concordance with our study. Truncated stop loss are indeed optimal when risk constraints are linked to a VaR or a CTE constraint. That is companies choose to let uninsured the worst possible states. Moreover the last numerical example (section 3.3) shows that full insurance above a deductible might be too costly to be realistic.

## 4.2 Alternative on Capital Market

Recently it has become widely appreciated that a single natural hazard could result in damages of several billions. For instance the total insured US catastrophic losses for 2005 are estimated to be more than \$50 billion, where the three major hurricanes Katrina, Rita and Wilma make up 90% of the total loss of the year<sup>12</sup>. Even if the insurance industry’s equity capital would be enough to absorb lots of catastrophic events, Cummins, Doherty and Lo (2002) explain that many insurers can become insolvent depending on the distribution of damage and their portfolio of policies. They note that in absence of costs, the Pareto optimal way to share risks is to mutualize all risks between all insurers.

The traditional instrument to spread risks between insurers is reinsurance. By reinsuring a layer of one line of business or of a specific risk, insurers buy and sell options on the loss index. Assuming risks can be measured in terms of an index, a temperature, a specified event, a wind speed, then a reinsurance arrangement can be interpreted as a portfolio of derivatives written on this underlying. For example, CAT Bonds are widely exchanged, if the defined catastrophic event does not occur, the investors receive their principal and interest equal to the risk-free rate plus a risk premium.

More precisely, Froot (2001) and Cummins, Lalonde and Phillips (2004) compare reinsurance layers with call spreads. They explain how insurers hedge their risks by forming a portfolio consisting in its losses (assume to be traded as a loss index  $X$ ) and a position in call option spreads on the loss index,

$$X - I_\theta(X) = X - \theta [(X - d)^+ - (X - l)^+] \quad (33)$$

where  $\theta$  corresponds to a coinsurance treaty. The indemnity is:

$$I_\theta(X) = \begin{cases} 0 & \text{If } X < d \\ \theta(X - d) & \text{If } X \in [d, l] \\ \theta(l - d) & \text{If } X > l \end{cases}$$

---

<sup>12</sup>Source: The annual report Guy Carpenter, “US Reinsurance Renewals at January 1, 2006”

In presence of regulators' minimum capital requirement and VaR or Conditional Tail expectation maximum risk exposure, we showed that the optimal reinsurance arrangement is (See formula (22)):

$$I^\alpha(X) = (X - d)\mathbf{1}_{X \in [d, q]}$$

where  $d$  is the deductible level and  $q$  the upper limit. Then,

$$I^\alpha(X) = (X - d)^+ - (X - q)^+ - (q - d)\mathbf{1}_{X > q}$$

which corresponds to a portfolio of derivatives, a long position on a call and a short position on a put and on a barrier bond (activated when the underlying  $X$  is above  $q$ ). Equivalent, the company possesses:

$$X - I^\alpha(X) = X - [(X - d)^+ - (X - q)^+ - (q - d)\mathbf{1}_{X > q}].$$

Comparing this portfolio with the call spread (33), we show it is optimal for companies to sell the bond corresponding to the right tail risk. They optimally give up the right tail, because there is no reason to pay for reinsurance after being ruined.

Using a simulation model, Cummins, Lalonde and Phillips (2004) study the efficiency of hedging with reinsurance or index-linked securities. They explain that reinsurance contracts are sold over the expected loss and that it is less efficient than hedging using contracts actuarially fair priced. Insurance-linked securities are mostly competitive with reinsurance in terms of price and hedge efficiency. Index-linked contracts can indeed be traded at significantly lower margins because of their liquidity and because they are less affected by moral hazard. Duplicate reinsurance arrangements on the market is a worth alternative risk transfer.

### 4.3 Extension to Random Maturity

So far, we have examined reinsurance arrangements in a one period model. The premium is paid at the beginning of the period and the indemnity is paid at the end of the period in case of a loss. In practice, indemnities are often paid upon an event if it occurs during the period of coverage. For instance the insurance company pays the premium  $\mu$  and receive the coverage  $I$  at time  $\tau$  when a pre-specified event occurs at time  $\tau$  during the coverage period<sup>13</sup>. We now extend our previous analysis to the random maturity situation.

At time 0, the initial wealth is  $W_0$  and a insurance premium  $\mu$  is paid. Let  $\tau$  be the triggered event and  $X_\tau$  the amount of the loss at that time. Then the indemnity  $I(X_\tau, \tau)$  is paid upon  $\tau$ . We assume the joint distribution of  $(X_\tau, \tau)$  is known. In case the event is observed, the final wealth at  $\tau$  is equal to ( $r$  is the interest rate):

$$W_\tau = (W_0 - \mu) e^{r\tau} - X_\tau + I(X_\tau, \tau)$$

---

<sup>13</sup>Examples include a default event, a catastrophic event, a terrorism event or an event that temperature below some level at some place at one given date.

Given  $\mu$ , its expected wealth is then equal to:

$$E[W_\tau] = (W_0 - \mu) E[e^{r\tau}] - E[X_\tau] + E[I(X_\tau, \tau)]$$

The premium principle is still based on the actuarial value  $\mu = (1 + \rho)E[I(X_\tau, \tau)]$ .

Regulators will ask companies to satisfy some constraints at any time and in particular at time  $\tau$ . The reinsurance market is then necessary to protect companies against unknown loss  $X_\tau$  at random time  $\tau$ . The optimization problem  $(\bar{A}_\beta)$  we consider earlier can be stated as follows:

$$(D) \quad \min_I \{P(W_\tau < W_0 - v)\} \quad s.t. \quad \begin{cases} 0 \leq I(x, \tau) \leq x \\ E[I(X_\tau, \tau)] = \beta \end{cases} \quad (34)$$

**Proposition 4.1** *Assuming  $(\tau, X_\tau)$  has a continuous cumulative distribution function,  $F(t, x) = P(\tau \leq t, X_\tau \leq x)$ , is strictly increasing with respect to both variables on  $[0, +\infty) \times [0, +\infty)$ ,  $\mu = (1 + \rho)\beta \leq v$ , and*

$$\beta \in (0, E[(X_\tau - (v - \mu)e^{r\tau})^+]), \quad (35)$$

*then there exists a positive  $\lambda_\beta$  such that the optimal coverage  $I^\tau$  solves the optimization program (D).*

$$I^\tau(X, t) = \begin{cases} 0 & \text{if } X \leq (v - \mu)e^{rt} \\ X - (v - \mu)e^{rt} & \text{if } (v - \mu)e^{rt} < X \leq (v - \mu)e^{rt} + \frac{1}{\lambda_\beta} \\ 0 & \text{if } X > (v - \mu)e^{rt} + \frac{1}{\lambda_\beta} \end{cases}$$

$\lambda_\beta$  is determined such that:

$$E[I^\tau(X, t)] = \beta.$$

The proof of this proposition is similar to the proof of Proposition 1.1 and the detail is thus omitted. The shape of the optimal contract is also similar to shape of  $I^{var}(X)$  and  $I^{cte}(X)$ . If the company purchases optimal reinsurance contract under VaR constraint, this reinsurance contract doesn't insure higher loss amount. The extension of the problem  $(B_\beta)$  and  $(C_\beta)$  to the random maturity framework is similar.

## 5 Conclusion

Risk management programs have been implemented by regulated insurance companies recently. It thus leads to an important problem for insurance company to maximize profit while consistent with a given risk management policy. We focus on primarily risk-neutral insurance companies subject to several risk measures imposed by regulators. We derive the design of the optimal reinsurance contract to maximize the profit, or equivalently to minimize the premium, when the risk management constraints are satisfied. We show that

the insurance companies have no incentive to protect themselves against extreme losses when regulatory requirements are based on Value-at-Risk or Conditional Tail Expectation maximum risk exposure. These results illustrate the prevalence of reinsurance contracts for risk neutral insurance companies and confirm observed behaviors of insurance companies that prefer to let uninsured the high layers of risks (Froot (2001)). We also found out that alternative risk management measure would lead insurance companies to fully hedge the right tail of the loss distribution.

In this paper we make model assumptions that there are no transaction cost for issuing reinsurance contract and purchasing contract, no background risk, and a single loss during the period of insurance protection. Moreover both issuer and issued are risk neutral, both parties know the probability distribution of the loss. Even though of the above mentioned model limitations, the results of this paper could be still used as “prototypes” by insurance company to design optimal risk management strategy, as well as regulators to impose appropriate risk measures. Because of the link and similarity between reinsurance market and capital market, our results also present alternative risk transfers mechanisms on a capital market.

# A Proofs

Recall that final wealth  $W$  is given by:

$$W = W_0 - \mu - X + I(X)$$

Then the event  $\{W \geq W_0 - v\}$  is equivalent to, in terms of the coverage  $I(X)$ ,

$$\{I(X) \geq \mu + X - v\}, \quad \mu = (1 + \rho)E[I(X)]$$

## A.1 Proposition 1.1

The problem  $(\bar{A}_\beta)$  could be reformulated as follows.

$$\max_I P(I(X) \geq \mu + X - v) \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] = \beta \end{cases}$$

**Lemma A.1** *If  $Y^*$  satisfies the three following properties:*

(i)  $0 \leq Y^* \leq X$ ,

(ii)  $E[Y^*] = \beta$ ,

(iii) *There exists a positive  $\lambda > 0$  such that for each  $\omega \in \Omega$ ,  $Y^*(\omega)$  is a solution of the following optimization problem:*

$$\max_{Y \in [0, X(\omega)]} \{\mathbb{1}_{\mu + X(\omega) - v \leq Y} - \lambda Y\}, \quad \mu = (1 + \rho)\beta$$

*then  $Y^*$  solves the current optimization problem  $(\bar{A}_\beta)$ .*

*Proof.* Given a coverage  $I$  which satisfies the constraints of the optimization program  $(\bar{A}_\beta)$ . Therefore, using (iii), we have,

$$\forall \omega \in \Omega, \quad \mathbb{1}_{\mu + X(\omega) - v \leq Y^*} - \lambda Y^* \geq \mathbb{1}_{\mu + X(\omega) - v \leq I(\omega)} - \lambda I(\omega)$$

Thus,

$$\mathbb{1}_{\mu + X(\omega) - v \leq Y^*(\omega)} - \mathbb{1}_{\mu + X(\omega) - v \leq I(\omega)} \geq \lambda (Y^*(\omega) - I(\omega))$$

We now take the expectation of the above inequality, therefore by condition (ii) one obtains,

$$P(\mu + X - v \leq Y^*) - P(\mu + X - v \leq I) \geq \lambda(\beta - E[I])$$

Therefore, applying the constraints of the variable  $I$ ,  $E[I(X)] = \beta$ ,

$$P(\mu + X - Y^* \geq v) - P(\mu + X - I \geq v) \geq 0$$

The proof of this lemma is completed. □

**Lemma A.2** When  $\mu = (1 + \rho)\beta$ ,  $\mu \leq v$ , each member of the following family  $\{Y_\lambda\}_{\lambda>0}$  satisfies the conditions (i) and (iii) of lemma A.1.

$$Y_\lambda(\omega) = \begin{cases} 0 & \text{if } X(\omega) < v - \mu \\ X(\omega) + \mu - v & \text{if } v - \mu \leq X(\omega) \leq v - \mu + \frac{1}{\lambda} \\ 0 & \text{if } X(\omega) > v - \mu + \frac{1}{\lambda} \end{cases}$$

*Proof.* The property (i) is obviously satisfied. Indeed we only study the case when  $v$  is more than the premium  $\mu$ , that is  $v \geq \mu$ .

First, if  $X(\omega) < v - \mu$ , then  $\mu + X(\omega) - v < 0$ , the function to maximize over  $[0, X(\omega)]$  is equal to  $1 - \lambda Y$ , decreasing over the interval  $[0, X(\omega)]$ , the maximum is thus obtained at  $Y^*(\omega) = 0$ .

Otherwise,  $X(\omega) \geq v - \mu$ . Since  $\mu \leq v$ , one has  $\mu + X(\omega) - v \leq X(\omega)$ . We consider two cases: firstly, if  $Y \in [0, \mu + X(\omega) - v]$ , then the function to maximize is  $-\lambda Y$ . It is decreasing with respect to the variable  $Y$ . Its maximum is 0, obtained at  $Y = 0$ . Secondly, if  $Y \in [\mu + X(\omega) - v, X(\omega)]$ , then the function to maximize is  $1 - \lambda Y$ . It is decreasing. Its maximum is obtained at  $Y = \mu + X(\omega) - v$  and its value is  $1 - \lambda(\mu + X(\omega) - v)$ . We compare the value  $1 - \lambda(\mu + X(\omega) - v)$  and 0 to decide whether the maximum is attained at  $Y = X(\omega) + \mu - v$  or  $Y = 0$ . Lemma A.2 is proved.  $\square$

**Proof of Proposition 1.1** Thanks to lemmas A.1 and A.2, it suffices to prove that there exists  $\lambda > 0$  such that  $Y_\lambda$  defined in lemma A.2 satisfies the condition (ii) of lemma A.1. We then compute its expectation.

$$\mathcal{E}_\lambda := E \left[ (X + \mu - v) \mathbf{1}_{X \in [v - \mu, v - \mu + \frac{1}{\lambda}]} \right]$$

It is obvious then:

$$\lim_{\lambda \rightarrow 0^+} \mathcal{E}_\lambda = E [(X - v + \mu)^+], \quad \lim_{\lambda \rightarrow +\infty} \mathcal{E}_\lambda = 0.$$

By Lebesgue dominance theorem we can easily prove the convergence property of  $\mathcal{E}_\lambda$  with respect to the parameter  $\lambda$ . Then the existence of a solution  $\lambda_\beta^* \in \mathbb{R}_+^*$  such that  $\mathcal{E}_\lambda = \beta$  comes from the assumption on the continuous distribution of  $X$  and thus the continuity of  $\mathcal{E}_\lambda$ . Thus we have proved the first part of this Proposition. The second part follows easily from the first part.  $\square$

## A.2 Proposition 3.1

Assuming  $\mu = (1 + \rho)\beta$ , the problem  $(\bar{B}_\beta)$  could be rewritten as:

$$\min_I (E [(\mu + X - I) \mathbf{1}_{\mu + X - I > v}]) \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E [I(X)] = \beta \end{cases}$$

**Lemma A.3** *If  $Y^*$  satisfies the three following properties:*

(i)  $0 \leq Y^* \leq X$ ,

(ii)  $E[Y^*] = \beta$ ,

(iii) *There exists a positive  $\lambda > 1$  such that for each  $\omega \in \Omega$ ,  $Y^*(\omega)$  is a solution of the following optimization problem:*

$$\min_{Y \in [0, X(\omega)]} \{(\mu + X - Y) \mathbb{1}_{Y < \mu + X(\omega) - v} + \lambda Y\}, \mu = (1 + \rho)\beta$$

*then  $Y^*$  solves the current optimization problem  $(\overline{B}_\beta)$ .*

*Proof.* The proof of lemma A.3 is similar to the proof of lemma A.1. We leave the details to the reader.  $\square$

**Proof of Proposition 3.1:** We use lemma A.3 and show that for  $\lambda > 1$ ,

$$Y_\lambda(\omega) = \begin{cases} 0 & \text{if } X(\omega) < v - \mu \\ X(\omega) + \mu - v & \text{if } v - \mu \leq X(\omega) \leq v - \mu + \frac{v}{\lambda - 1} \\ 0 & \text{if } X(\omega) > v - \mu + \frac{v}{\lambda - 1} \end{cases}$$

satisfies conditions (i) and (iii) of lemma A.3. If  $X(\omega) + \mu - v < 0$  then  $Y^* = 0$ . Otherwise  $0 \leq \mu + X(\omega) - v < X$ . Similar to the proof of Proposition 1.1 we can prove that  $Y = \mu + X(\omega) - v$  is the maximum one if  $v - \mu \leq X(\omega) \leq v - \mu + \frac{v}{\lambda - 1}$ , else the maximum one is  $Y = 0$  if  $X(\omega) > v - \mu + \frac{v}{\lambda - 1}$ .

We then compute its expectation.

$$\mathcal{E}_\lambda := E \left[ (X + \mu - v) \mathbb{1}_{X \in (v - \mu, v - \mu + \frac{v}{\lambda - 1})} \right]$$

It is obvious then:

$$\lim_{\lambda \rightarrow 1^+} \mathcal{E}_\lambda = E[(X - v + \mu)^+], \quad \lim_{\lambda \rightarrow +\infty} \mathcal{E}_\lambda = 0.$$

The existence of a solution  $\lambda^* > 1$  such that  $\mathcal{E}_\lambda = \beta$  comes from the assumption on the continuous distribution of  $X$  and thus the continuity of  $\mathcal{E}_\lambda$ . Thus we have proved the first part of Proposition 3.1. The second part follows easily from the first part.  $\square$

### A.3 Proposition 3.2

**Lemma A.4** *If  $Y^*$  satisfies the three following properties:*

(i)  $0 \leq Y^* \leq X$ ,

(ii)  $E[Y^*] = \beta$ ,

(iii) *There exists a positive  $\lambda > 0$  such that for each  $\omega \in \Omega$ ,  $Y^*(\omega)$  is a solution of the following optimization problem:*

$$\min_{Y \in [0, X(\omega)]} \{(\mu + X(\omega) - Y - v)^2 \mathbb{1}_{Y < \mu + X(\omega) - v} + \lambda Y\}, \mu = (1 + \rho)\beta$$

*then  $Y^*$  solves the optimization problem  $(\overline{C}_\beta)$ .*

*Proof.* Let  $Y^*$  be a random variable satisfying the three above conditions of the lemma. On the other hand, given another available payoff  $I$  which satisfies the constraints of the above optimization program. Therefore, using (iii), we have,  $\forall \omega \in \Omega$ ,

$$\begin{aligned} & (\mu + X(\omega) - Y^*(\omega) - v)^2 \mathbb{1}_{Y^*(\omega) < \mu + X(\omega) - v} + \lambda Y^*(\omega) \leq \\ & (\mu + X(\omega) - I(\omega) - v)^2 \mathbb{1}_{I(\omega) < \mu + X(\omega) - v} + \lambda I(\omega) \end{aligned}$$

Thus,

$$\begin{aligned} & (\mu + X(\omega) - Y^*(\omega) - v)^2 \mathbb{1}_{Y^*(\omega) < \mu + X(\omega) - v} - (\mu + X(\omega) - I(\omega) - v)^2 \mathbb{1}_{I(\omega) < \mu + X(\omega) - v} \\ & \leq \lambda (I(\omega) - Y^*(\omega)) \end{aligned}$$

We now take the expectation of the above inequality, therefore by condition (ii) one obtains,

$$E [(\mu + X - Y^* - v)^2 \mathbb{1}_{Y^* < \mu + X - v}] - E [(\mu + X - I - v)^2 \mathbb{1}_{I < \mu + X - v}] \leq \lambda (E[I] - \beta)$$

Therefore, applying the constraints of the variable  $I$ ,  $E[I(X)] = \beta$ ,

$$E [(\mu + X - Y^* - v)^2 \mathbb{1}_{Y^* < \mu + X - v}] \leq E [(\mu + X - I - v)^2 \mathbb{1}_{I < \mu + X - v}]$$

The proof of this lemma is completed. □

**Lemma A.5** *When  $\mu \leq v$ , each member of the following family  $\{Y_\lambda\}_{\lambda > 0}$  satisfies the conditions (i) and (iii) of lemma A.4.*

$$Y_\lambda(\omega) = \begin{cases} 0 & \text{if } X(\omega) < v - \mu + \frac{\lambda}{2} \\ X(\omega) + \mu - v - \frac{\lambda}{2} & \text{if } v - \mu + \frac{\lambda}{2} \leq X(\omega) \end{cases}$$

*Proof.* The property (i) is obviously satisfied. We now prove the property (iii).

First, if  $X(\omega) < v - \mu$ , then  $\mu + X(\omega) - v < 0$ , the function to minimize over  $[0, X(\omega)]$  is equal to  $\lambda Y$ , increasing over the interval  $[0, X(\omega)]$ , the minimum is thus obtained at  $Y^*(\omega) = 0$ .

Otherwise,  $X(\omega) \geq v - \mu$ . Since  $\mu \leq v$ , one has  $\mu + X(\omega) - v \leq X(\omega)$ . Thus we have to solve the optimization program under the assumption  $0 \leq \mu + X(\omega) - v \leq X(\omega)$ . There are two cases: firstly, if  $Y \in [0, \mu + X(\omega) - v]$ , then the function to minimize is

$$\phi_1(Y) = (\mu + X(\omega) - v - Y)^2 + \lambda Y.$$

Its minimum is  $\max(0, X(\omega) + \mu - v - \frac{\lambda}{2})$ . Secondly, if  $Y \in [\mu + X(\omega) - v, X(\omega)]$ , then the function to minimize is

$$\phi_2(Y) = \lambda Y.$$

Its minimum is obtained at  $Y = \mu + X(\omega) - v$  and its value is  $\lambda(\mu + X(\omega) - v)$ . We then compare this value with the previous minimum:

- When  $0 < X(\omega) + \mu - v - \frac{\lambda}{2}$ ,  $\Phi_1\left(X(\omega) + \mu - v - \frac{\lambda}{2}\right) = \Phi_2(\mu + X(\omega) - v) - \frac{\lambda^2}{4} < \Phi_2(\mu + X(\omega) - v)$ .
- When  $0 > X(\omega) + \mu - v - \frac{\lambda}{2}$ ,  $\Phi_1(0) = (\mu + X(\omega) - v)^2$ . Since  $\frac{\lambda}{2} > X(\omega) + \mu - v$ ,  $\Phi_1(0) < \frac{\Phi_2(\mu + X(\omega) - v)}{2} < \Phi_2(\mu + X(\omega) - v)$ .

Obviously, the minimum is thus obtained when  $Y = \max\left(0, X(\omega) + \mu - v - \frac{\lambda}{2}\right)$ . Lemma A.5 is proved.  $\square$

**Proof of Proposition 3.2.** Thanks to lemmas A.4 and A.5, one only has to prove that there exists  $\lambda > 0$  such that  $Y_\lambda$  defined in lemma A.5 satisfies the condition (ii) of lemma A.4. We then compute its expectation.

$$\mathcal{E}_\lambda := E \left[ \left( X + \mu - v - \frac{\lambda}{2} \right) \mathbb{1}_{X \in \left[ v - \mu + \frac{\lambda}{2}, +\infty \right)} \right]$$

It is obvious then:

$$\lim_{\lambda \rightarrow 0^+} \mathcal{E}_\lambda = E \left[ (X - v + \mu)^+ \right], \quad \lim_{\lambda \rightarrow +\infty} \mathcal{E}_\lambda = 0.$$

The existence of a solution  $\lambda^* \in \mathbb{R}_+^*$  such that  $\mathcal{E}_\lambda = \beta$  comes from the assumption on the continuous distribution of  $X$  and thus the continuity of  $\mathcal{E}_\lambda$ . Thus we have proved the first part of this Proposition. The second part follows easily from the first part.

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