Anomalies in Intertemporal Choice?*

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Abstract

This paper argues that observations of non-stationary choice behavior need not necessarily imply specific properties of the individual's discount function. As we show, the observed "anomalies" in intertemporal choice can alternatively be explained by an individual's perception of the risk that is involved whenever an outcome is to be received in the future. This risk may concern the size of the actual outcome or the endowment consumption stream to which the outcome is added. Both types of uncertainty naturally appear in the context of intertemporal choice and both are difficult to control in experiments. We show how relative degrees of changes in risk over time can predict choices.

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JEL classification: D91, D81

1 Introduction

Individuals generally value immediate payoffs higher than later payoffs, i.e. they discount future payoffs. There are many reasons for discounting future outcomes (Frederick, Loewenstein, and O'Donoghue, 2002). One of them is that the future is risky and that there is

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a positive probability that we will no longer be alive in the future. In this paper we will not focus on the fact that risk causes impatience as such. We will rather study how the interaction of risk and impatience influences choice behavior over time. More specifically, we will analyze the extent to which we can draw conclusions about the curvature of an individual's discount function merely from observing choice behavior. Keren and Roelofsma (1995) and Benzion, Rapoport, and Yagil (1989) discussed before that risk may be underlying some of the findings of non-constant discounting. The second half of this paper formalizes their ideas.

A widely used model in intertemporal choice that captures an individual's impatience is discounted utility. For a long time Samuelson's (1937) constant, that is exponential, discounting has been the most popular model for decision making in economics. Under constant discounting preferences satisfy stationarity, i.e. the preference between two streams of outcomes does not change if the delivery of every outcome in both streams is postponed by a common delay. Under some additional assumptions stationarity implies time-consistent behavior. Strotz (1956) was the first to analyze a formal economic model with time-inconsistent preferences. Since then many psychological and economic studies have found evidence against stationary preferences (a.o. Benzion, Rapoport, and Yagil, 1989; Bleichrodt and Johannesson, 2001; Cairns and van der Pol, 2000; Green, Fristoe, and Myerson, 1994; Kirby and Marakovic, 1995; Mazur, 1987, 2001; Read and Read, 2004; Rodriguez and Logue, 1988; and Thaler, 1981). Most studies found decreasing impatience, as in Thaler's (1981) example of a person who prefers one apple today over two apples tomorrow, but at the same time prefers two apples in 51 days over one apple in 50 days. Loewenstein and Prelec (1992) called this the "common difference effect." As a consequence of the empirical evidence in favor of decreasing impatience, quasi-hyperbolic (Phelps and Pollak, 1968) and generalized hyperbolic discounting (Loewenstein and Prelec, 1992) were introduced. These discount functions satisfy decreasing impatience. Hyperbolic discounting models are becoming increasingly popular in applications (Angeletos et al., 2001; Barro 1999; Harris and Laibson, 2001; Krusell and Smith, 2003; Laibson, 1997; Luttmer and Mariotti, 2003, 2006; O'Donoghue and Rabin, 1999a, 1999b; and Thaler and Benartzi, 2004).

In addition to non-stationarities, other anomalies in intertemporal choice have been reported in the literature.[1] The magnitude effect refers to the finding that large gains suffer less proportional discounting than small ones. The gain-loss asymmetry is the property that losses are discounted at a lower rate than gains are. Finally, the delay-speedup asymmetry refers to asymmetric preferences between delaying and speeding up the receipt of outcomes. We do not discuss this asymmetry here, because it is a mere framing effect (Loewenstein and Prelec, 1992) and can therefore not be explained by any normative theory, in particular not by discounted expected utility. The anomalies all seem to contradict the traditional theory of Samuelson and decision models have been developed to incorporate them (Baucells and Heukamp, 2007; Ok and Masatlioglu, 2003).

This paper shows that non-stationarity, the magnitude effect and the gain-loss asymmetry may be less of an anomaly than has often been thought and instead can be fully compatible with Samuelson's (1937) constant discounting. The magnitude effect can be explained by merely concave utility, rather than the often assumed linearity. This also holds for the gain-loss asymmetry, as was pointed out before by Loewenstein and Prelec (1992). Furthermore and most importantly we will show that non-stationary behavior occurs when subjects perceive their endowments or outcomes as risky. Both types of risk naturally appear in the context of intertemporal choice. When choosing between two different outcomes the decision maker (DM) determines his preference between the consumption streams that result if the outcomes are added to his endowment consumption. Since the latter involves future consumption, it seems natural to assume that the DM considers the endowment to be subject to risk.[2] At the same time any outcome that is to be received in the future may itself be considered risky. Whoever has promised to deliver the outcome may fail on the obligation. Whether the DM takes into account outcome risk depends, among other things, on the reputation of the individual or institution who has promised the outcome and on the precautions that have been taken in order to guarantee its delivery.

[1] For an extensive survey see Frederick, Loewenstein, and O'Donoghue (2002).

[2] The role of this background risk for decisions making under risk has also been studied by Gollier and Pratt (1996) and Eeckhoudt et al. (1996).

In general both endowments and outcomes will be risky. However, in order to study the influence of each type of risk on the DM's choice behavior we analyze both types of risk in isolation. We show that for a DM who satisfies constant discounting observed behavior may suggest increasing, or decreasing impatience, respectively, whenever the increase in endowment risk in the near future is relatively large, or small, compared to the change in endowment risk in the far future. Similarly, when outcome-risk increases relatively strongly in the near, or, respectively, far, future compared to in the far, or near, future, the DM will seem to have decreasing, or increasing impatience.

An interpretation of our results is that in order to draw the right conclusions from empirical tests of discounting, it is important to have control over the subjects' perception of the risk that is involved, since this has a strong influence on choice behavior. One way to control the subjects' perception of risk is to restrict to time frames where the change in risk of the subjects concerning their future endowment or future outcomes does not change too much.

This paper is related to Noor (2007), who shows that under constant discounting all anomalies in intertemporal choice, except for intransitivity, can be explained by changes in future endowments. The papers are complementary in the sense that Noor rationalizes observed behavior by perceived deterministic increases in future endowments while we rationalize behavior by perceived risk in future endowments or outcomes. Moreover, we show that all kinds of preference reversals can be obtained for a discount function satisfying increasing, constant or decreasing impatience even if we take the decision maker's utility function as given and only vary the perceived risk.

Several recent studies also analyze the effect of risk or uncertainty on intertemporal choice and its implications on the anomalies often found. Dasgupta and Maskin (2005) assume that the timing of outcomes is risky. Bommier (2006) considers uncertainty about lifetime. Boyarchenko and Levendorskii (2006) assume that outcomes are risky. The decision makers in their study pay, or receive, a particular amount at a particular date t in order to receive, or pay, an outcome T dates later. The decision maker can choose date t. Thus, the receipt of the outcome T dates later in return for a payment today is seen as an option which can be exercised at any date t either with or without a limit. The authors use option pricing theory to show that this setting produces the common intertemporal choice anomalies, even though the underlying discounting is constant. The main differences between their approach and ours is that they use the theory of real options, which we do not, and that they do not make the distinction between outcome and endowment risk, which we do.

Finally, apart from many studies that find decreasing impatience, there are also studies that do not find such evidence (Read et al., 2005; Rubinstein, 2003). Some even find increasing impatience (Attema et al., 2006; Chesson and Viscusi, 2003; Frederick, 1999; Gigliotti and Sopher, 2003; Onay and Oncüler, 2007; Read, Airoldi and Loewe, 2005; Sayman and Oncüler, 2006). Our results provide a simple and intuitive explanation for increasing impatience. Also, we show how decreasing, constant, and increasing impatience can coexist in a natural and simple model. Thus, the studies finding increasing impatience do not necessarily contradict the studies that found decreasing impatience.

This paper is organized as follows. Section 2 introduces the general setting that we consider and the assumptions made. The next two sections consider the case of risky endowments (Section 3) and the case of risky outcomes (Section 4). Section 5 discusses the magnitude effect and the gain-loss asymmetry. Finally, Section 6 concludes. All proofs are in the appendix.

2 The setting

This section introduces basic notation and concepts. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$. The set $\mathcal{R}$ is the set of all random variables with realizations in $\mathbb{R}$ and with finite expectation. $\mathcal{R}^k$ contains all k-tuples of independently distributed random variables. For two random variables $x, y \in \mathcal{R}$, $x =_d y$ means that x has the same distribution as y.

We consider a decision-maker (DM) who has a lifetime or planning horizon of T periods. Date $t = 0$ corresponds to 'today' and date T is the final period. The DM has a complete and transitive preference relation $\succcurlyeq$ on $\mathcal{R}^{T+1}$. He has an *endowment* $\omega = (\omega_0, \ldots, \omega_T) \in$

$\mathcal{R}^{T+1}$. A typical example for a DM's endowment is his current and future income. A different example arises if the DM only evaluates deviations from some reference income as in Prospect Theory (Kahneman and Tversky, 1979). In this case the DM's endowment in any period is given by the deviation of his actual income from the reference income.

We assume that date-0-endowment is riskless, i.e. $Var(\omega_0) = 0$, whereas date-t-endowment ω_t in general will be a random variable.[3] The endowment ω_t is *riskless*, whenever $Var(\omega_t) = 0$ and it is *risky* otherwise. If ω_t is riskless we identify ω_t with $E(\omega_t) \in \mathbb{R}$. *Endowment risk* is *i.i.d.* if endowments for all future dates $t > 0$ are independently and identically distributed. Endowment $\omega_{\bar{t}}$ is a *mean-preserving spread* of ω_t if $\omega_{\bar{t}} =_d \omega_t + \varepsilon$ and $\omega_{\bar{t}} \neq_d \omega_t$, with $\varepsilon \in \mathcal{R}$ and $E(\omega_{\bar{t}} \,|\, \omega_t = w_t) = w_t$ for all realizations w_t of ω_t. We say that *endowment risk is increasing* if $\omega_{\bar{t}}$ is a mean-preserving spread of ω_t for every $0 \le t < \bar{t} \le T$.

Throughout this paper a DM will have a choice between receiving one *outcome* $x \in \mathcal{R}$ at a particular date or another outcome $y \in \mathcal{R}$ at another date. We use the notation $(t : x)$ to indicate that on top of the endowment, the DM receives an outcome $x \in \mathcal{R}$ at time $t \in \mathcal{T} = \{0, \ldots, T\}$, where x can be riskless or risky and is always independently distributed from endowments. We call $(t : x)$ a *dated outcome*. We assume that the outcome x is consumed at the date when it is received. Thus, a DM who receives $(t : x)$ consumes the *consumption stream* $\xi^{t,x} = (\xi_0^{t,x}, \xi_1^{t,x}, \ldots, \xi_T^{t,x}) \in \mathcal{R}^{T+1}$ with $\xi_t^{t,x} = \omega_t + x$ and $\xi_{t'}^{t,x} = \omega_{t'}$ for all $t' \neq t$. Let $\mathcal{X}$ be the set of all dated outcomes $(t : x)$ such that $t \in \mathcal{T}$, $x \in \mathcal{R}$ and x and ω_t are independently distributed. The DM's preference relation $\succcurlyeq$ on consumption streams defines an *induced preference* relation $\succcurlyeq^*$ on $\mathcal{X}$ as follows. Let $(t : x), (t' : y) \in \mathcal{X}$. Then

$$(t : x) \succcurlyeq^* (t' : y) \quad \text{if and only if} \quad \xi^{t,x} \succcurlyeq \xi^{t',y}.$$

Thus, $(t : x) \succcurlyeq^* (t' : y)$ means that the DM weakly prefers the consumption stream where on top of his endowment he receives x at date t, over the consumption stream where on top of his endowment he receives y at date t'. Observe that $\succcurlyeq^*$ is complete and transitive,

[3] We use the notation Var to denote the variance and E to denote the expectation of a random variable.

because $\succcurlyeq$ has these properties. This paper addresses the question to what extent we can draw conclusions about preferences $\succcurlyeq$ from observing induced preferences $\succcurlyeq^*$.

We assume throughout that the DM satisfies *discounted expected utility*, i.e. $\succcurlyeq$ can be represented by a *discounted expected utility function*:

$$\xi \succcurlyeq \xi' \iff \sum_{t=0}^{T} \delta(t) EU(\xi_t) \geq \sum_{t=0}^{T} \delta(t) EU(\xi'_t),$$

for $\xi, \xi' \in \mathcal{R}^{T+1}$, where δ is a *discount function* and EU stands for the *expected utility* associated with a *von Neumann-Morgenstern utility function* $U : \mathbb{R} \to \mathbb{R}$. We assume that the utility function U is three times continuously differentiable, increasing ($U' > 0$) and strictly concave ($U'' < 0$). Moreover, we assume that U satisfies strictly decreasing absolute risk aversion, i.e. $-U''(x)/U'(x)$ is strictly decreasing in x, which implies that $U''' > 0$. The discount function δ is assumed to be decreasing in t ($\delta(\bar{t}) < \delta(t)$ for $\bar{t} > t$), and to satisfy $\delta(t) > 0$ for all $t \in \mathcal{T}$, and $\delta(0) = 1$. The discount function satisfies decreasing (constant, increasing) impatience if outcomes that are received in the far future are discounted less (equally, more) than outcomes that are received in the near future. This is formalized in the following definition which is equivalent to Prelec's (2004) definition.

Definition 2.1

The discount function δ satisfies decreasing (constant, increasing) impatience if for all $t, \bar{t}$ with $t < \bar{t}$ and for all $\tau > 0$

$$\frac{\delta(t)}{\delta(t+\tau)} \; > \; (=, <) \; \frac{\delta(\bar{t})}{\delta(\bar{t}+\tau)}$$

Constant impatience is equivalent to Samuelson's (1937) constant discounting: there is a constant $\phi < 1$ such that $\delta(t) = \phi^t$ for every t.

Let $x \in \mathbb{R}$ and $\varepsilon \in \mathcal{R}$ be such that $EU(x + \varepsilon)$ exists and is finite. Then by $\pi(x, \varepsilon)$ we denote the *risk premium* for the risk ε at x, i.e. $\pi(x, \varepsilon)$ is defined by $EU(x + \varepsilon) = U(x + E(\varepsilon) - \pi(x, \varepsilon))$. Utility U satisfies strictly decreasing absolute risk aversion if and only if $\pi(x, \varepsilon)$ is strictly decreasing in x for all $\varepsilon \in \mathcal{R}$, whenever the distribution of ε is nondegenerate, i.e. whenever $\text{Prob}(\varepsilon = E(\varepsilon)) < 1$ (Pratt, 1964).

Throughout the paper we assume that at every date the expected utility of the endowment exists and is finite.

Assumption I For all t, $|EU(\omega_t)| < \infty$.

Assumption I in particular holds for all endowments, where ω_t has a bounded support for all t.

It is assumed throughout that utility U is known and that the discount function is not known. Imagine, for instance, an experimenter who elicited the utility function from the DM through the observation of choices between outcomes that are received immediately, i.e. at date 0. Now, by observing choices between dated outcomes, i.e. by observing induced preferences, he wants to elicit the DM's discount function under the assumption that the DM has a discounted expected utility function. Under discounted expected utility $(t : x) \succcurlyeq^* (t' : y)$ if and only if

$$\delta(t)\left[EU(\omega_t + x) - EU(\omega_t)\right] \geq \delta(t')\left[EU(\omega_{t'} + y) - EU(\omega_{t'})\right]. \tag{1}$$

We say that the induced preference $(t : x) \succcurlyeq^* (t' : y)$ can be *supported by* a discount function δ if (1) holds for that particular δ.

3 Choices under endowment risk

This section will show that induced preferences that, at first sight, suggest increasing (decreasing) impatience do not rule out constant or decreasing (increasing) impatience. The results are driven by a change in endowment risk over time. In order to focus on the effect of endowment risk we only consider riskless outcomes in this section. As we will show, a choice pattern suggesting increasing impatience can be observed for a discounted expected utility function with constant or decreasing impatience as long as the change in endowment risk in the near future is relatively large compared to the change in risk in the far future. Similarly, a choice pattern that suggests decreasing impatience can be observed for a discounted expected utility function with constant or increasing impatience when the change in endowment risk in the near future is small compared to that change in the far future. Hence, since perceived endowment risk is unobservable, one has to be careful when drawing conclusions about preferences from observed behavior in intertemporal choice. In experiments it may therefore be instructive to elicit subjects' perceived change in endowment risk over time.

We start by considering an i-reversal. An i-reversal is an induced preference between two dated outcomes that seems to suggest increasing impatience. We discuss i-reversals first, because we will start with a setting that makes it easiest to grasp the intuition how risk concerning endowments influences decisions.

Definition 3.1 ***i-reversal***

Induced preferences $\succcurlyeq^*$ exhibit an i-reversal at the outcomes $y > x > 0$ and dates $\bar{t} > t \geq 0$ and $\tau > 0$ if

$$(t : x) \sim^* (t + \tau : y) \ \ \& \ \ (\bar{t} : x) \succ^* (\bar{t} + \tau : y).$$

An i-reversal seems to suggest increasing impatience: the DM is indifferent between receiving x at the early date t and receiving y, τ periods later, but he is no longer willing to wait for y if the delay is from the later date $\bar{t}$ to $\bar{t} + \tau$. Thus, the DM appears to be more impatient in the far future than in the near future.

If the DM keeps using the same procedure to evaluate outcome streams throughout time, then an i-reversal implies a time-inconsistency. To see this suppose that a DM exhibits an i-reversal at the outcomes $y > x > 0$ and dates $\bar{t} > t \geq 0$ and $\tau > 0$. Then, by monotonicity and continuity of preferences, we can find an outcome $\tilde{x}$ such that

$$(t : \tilde{x}) \prec^* (t + \tau : y) \ \ \& \ \ (\bar{t} : \tilde{x}) \succ^* (\bar{t} + \tau : y).$$

Hence, if today the DM has the choice between $(\bar{t} : \tilde{x})$ and $(\bar{t} + \tau : y)$, he will choose $(\bar{t} : \tilde{x})$. In $\bar{t} - t$ days these two options will be $(t : \tilde{x})$ and $(t + \tau : y)$. Thus, if the realized date-$(\bar{t} - t)$-endowment is close to ω_0, then in $\bar{t} - t$ days the DM will revise his choice and choose $(t + \tau : y)$.

In a context where perceived endowment risk is changing over time this apparent preference reversal, however, is not a proof of inconsistent behavior on the part of the DM. This is due to the fact that the choice between $(\bar{t} : \tilde{x})$ and $(\bar{t} + \tau : y)$ is not the same as the choice between $(t : \tilde{x})$ and $(t + \tau : y)$, $\bar{t} - t$ periods later if the endowment risk the DM perceives today differs from the endowment risk he perceives at $\bar{t} - t$.

As we will show next i-reversals in fact do not rule out constant or decreasing impatience. In particular, an i-reversal does not necessarily contradict Samuelson's (1937) constant discounting. To give an intuition for this result observe that obtaining a positive payoff on top of a risky endowment has two effects. First, it increases wealth at the given date. Second, the DM can cope better with the risk involved in consumption at that date, because at a higher wealth level he is less risk averse due to strictly decreasing absolute risk aversion. Hence, upon receiving a positive and riskless payoff the DM suffers less from the uncertainty involved in his endowment. The larger the increase in risk over time, the more valuable it is to delay the receipt of an outcome. Suppose that the increase in endowment risk is larger in the near than in the far future. Then, it is relatively more valuable to delay the receipt of an outcome in the near than in the far future. Hence, if the DM is indifferent between receiving outcome x at date t and receiving outcome y at date $t + \tau$, then at date $\bar{t} > t$, when the increase in risk is smaller, waiting for outcome y becomes less valuable, because the risk to be compensated is less. This explains an i-reversal.

We first consider a DM with i.i.d. endowment risk. The next theorem shows that an i-reversal does not rule out constant or decreasing impatience.

Theorem 3.2 *Consider a DM with i.i.d. endowment risk and let* $\omega_t = \bar{\omega}$ *for all* $t \geq 1$ *for some* $\bar{\omega} \in \mathcal{R}$*. Assume that* $U(\omega_0) > EU(\bar{\omega})$ *and let the DM satisfy an i-reversal at date 0, i.e.*

$$(0 : x) \sim^* (\tau : y) \;\; \& \;\; (\bar{t} : x) \succ^* (\bar{t} + \tau : y),$$

where $y > x > 0$ *and* $\bar{t}, \tau > 0$*, suggesting increasing impatience. This reversal can be supported by a discount function satisfying increasing, constant, or decreasing impatience; in particular it can be supported by a generalized hyperbolic discount function* $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$ *for some* $\alpha, \beta > 0$.[4]

The next theorem makes behavioral predictions, i.e. it predicts i-reversals. If the DM is indifferent between receiving x today and y at a later date τ, then, if endowment risk is i.i.d., increasing and constant impatience imply an i-reversal for any $t > 0$. Thus, if the receipt of x and y is delayed by any amount of time t, then the DM will strictly prefer the earlier outcome. A slightly weaker result holds if the discount function is of the generalized hyperbolic form, i.e. $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$, with $\alpha, \beta > 0$. In this case there will be an i-reversal for a given delay t whenever α is sufficiently small. Moreover, if the DM has a generalized hyperbolic discount function and if there is an i-reversal for some delay $\bar{t}$, then any delay $t < \bar{t}$ also results in a strict preference for the earlier outcome. Thus, if we observe one i-reversal, then we will observe many more i-reversals.

Theorem 3.3 *Consider a decision maker with i.i.d. endowment-risk and let* $\omega_t = \bar{\omega}$ *for all* $t \geq 1$ *for some* $\bar{\omega} \in \mathcal{R}$*. Assume that* $U(\omega_0) > EU(\bar{\omega})$ *and* $(0 : x) \sim^* (\tau : y)$ *where* $y > x > 0$ *and* $\tau > 0$*.*

(*i*) *If the DM's discount function satisfies increasing or constant impatience, then* $(t : x) \succ^* (t + \tau : y)$ *for all* $t > 0$*.*

(*ii*) *For every* $t > 0$ *there is an* A *such that if the DM has a generalized hyperbolic discount function* $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$*, with* $\alpha, \beta > 0$*, i.e. the discount function satisfies decreasing impatience, and if* $\alpha < A$ *then* $(t : x) \succ^* (t + \tau : y)$*.*

[4]The generalized hyperbolic discount function was proposed by Loewenstein and Prelec (1992).

(*iii*) *If the DM's discount function is given by* $\delta(t) = (1+\alpha t)^{-\beta/\alpha}$, *where* $\alpha, \beta > 0$, *and if* $(\bar{t} : x) \succ^* (\bar{t} + \tau : y)$, *then* $(t : x) \succ^* (t + \tau : y)$ *for all* t *with* $0 < t \leq \bar{t}$.

Theorems 3.2 and 3.3 are driven by the fact that today the increase in endowment risk is larger than in the future. Since we assumed i.i.d. endowment risk, under constant or decreasing impatience an i-reversal can only occur for $t = 0$. As we will show next, i-reversals can also occur for $t > 0$ if endowment risk is increasing.

Theorem 3.4 *Let the DM satisfy an i-reversal, i.e.*

$$(t : x) \sim^* (t + \tau : y) \;\&\; (\bar{t} : x) \succ^* (\bar{t} + \tau : y),$$

where $y > x > 0$, $\bar{t} > t \geq 0$, $\tau > 0$. *Then there exist endowments with increasing endowment risk, such that the reversal can be supported by a discount function satisfying increasing, constant, or decreasing impatience; in particular it can be supported by a generalized hyperbolic discount function* $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$ *for some* $\alpha, \beta > 0$.

The next theorem provides conditions for i-reversals to occur under increasing, constant or decreasing impatience and increasing endowment risk. The crucial assumption is that the increase in endowment risk in the near future is relatively large compared to the increase in risk in the far future.

Theorem 3.5 *Let* $y > x > 0$, $\bar{t} > t \geq 0$, *and* $\tau > 0$. *Let endowment risk be increasing. For every* M *there is an* N *such that if*

$$\begin{aligned} \left| [EU(\omega_{\bar{t}} + x) - EU(\omega_{\bar{t}})] - [EU(\omega_t + x) - EU(\omega_t)] \right| &> M \text{ and} \\ \left| [EU(\omega_{\bar{t}+\tau} + y) - EU(\omega_{\bar{t}+\tau})] - [EU(\omega_{t+\tau} + y) - EU(\omega_{t+\tau})] \right| &< N \end{aligned}$$

and if

$$(t : x) \sim^* (t + \tau : y)$$

then the following holds.

(i) If the DM's discount function satisfies increasing or constant impatience then

$$(\bar{t} : x) \succ^* (\bar{t} + \tau : y).$$

(ii) There is an A such that if the DM has a generalized hyperbolic discount function $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$ with $\alpha, \beta > 0$ and $\alpha < A$ then

$$(\bar{t} : x) \succ^* (\bar{t} + \tau : y).$$

The previous results have shown that even though an i-reversal suggests increasing impatience, it does not rule out a discount function that exhibits constant or decreasing impatience, depending on the relative change in risk. If the increase in endowment risk in the near future is large compared to the increase in risk in the far future, then i-reversals can be observed for all types of discount functions. We will now consider a d-reversal, which suggests decreasing impatience. As we will show, a d-reversal does not rule out a discount function that satisfies constant or increasing impatience.

Definition 3.6 ***d-reversal***
Induced preferences $\succcurlyeq^*$ exhibit a d-reversal at the riskless outcomes $y > x > 0$ and dates $\bar{t} > t \geq 0$ and $\tau > 0$ if

$$(t : x) \sim^* (t + \tau : y) \;\;\&\;\; (\bar{t} : x) \prec^* (\bar{t} + \tau : y).$$

A d-reversal suggests decreasing impatience. The DM is indifferent between receiving x at date t and waiting for the larger outcome y to be received at date $t + \tau$, but once the receipt of both outcomes is delayed by the same amount of time, the DM prefers to wait for the better outcome. Nevertheless, the next theorem will show that a d-reversal can also be observed when the discount function satisfies constant or increasing impatience.

Theorem 3.7 *Let the DM satisfy a d-reversal, i.e.*

$$(t : x) \sim^* (t + \tau : y) \ \& \ (\bar{t} : x) \prec^* (\bar{t} + \tau : y)$$

where $y > x > 0$, *and* $\bar{t} > t \geq 0$, $\tau > 0$. *Then there exist endowments with increasing endowment risk, such that the reversal can be supported by a discount function satisfying decreasing, constant or increasing impatience; in particular it can be supported by a discount function* $\delta(t) = \exp(-bt^{1+g})$ *for some* $b, g > 0$.

As long as the increase in endowment risk in the near future is small compared to the increase in risk in the far future d-reversals occur under decreasing, constant or increasing impatience as we show in the following theorem.

Theorem 3.8 *Let* $y > x > 0$, $\bar{t} > t \geq 0$, *and* $\tau > 0$. *Let endowment risk be increasing. Then, for every* M *there is an* N *such that if*

$$\begin{aligned} |\,(EU(\omega_{\bar{t}+\tau} + y) - EU(\omega_{\bar{t}+\tau})) - (EU(\omega_{t+\tau} + y) - EU(\omega_{t+\tau}))\,| &> M \text{ and} \\ |\,(EU(\omega_{\bar{t}} + x) - EU(\omega_{\bar{t}})) - (EU(\omega_t + x) - EU(\omega_t))\,| &< N \end{aligned}$$

and if

$$(t : x) \sim^* (t + \tau : y)$$

then the following holds.

(i) If the DM's discount function satisfies decreasing or constant impatience then

$$(\bar{t} : x) \prec^* (\bar{t} + \tau : y).$$

(ii) There is a G *such that if the DM has a discount function* $\delta(t) = \exp(-bt^{1+g})$ *with* $b, g > 0$, *i.e. the discount function satisfies increasing impatience, and if* $g < G$ *then*

$$(\bar{t} : x) \prec^* (\bar{t} + \tau : y).$$

4 Choices under outcome risk

While in the previous section we only considered riskless outcomes, in this section we address the case of risky outcomes. This section formalizes the ideas of Keren and Roelofsma (1995). We assume that the DM perceives any outcome x that is received at some date $t > 0$ to be risky, i.e. the DM perceives the outcome to be $x + \varepsilon_{x,t}$ for some $\varepsilon_{x,t} \in \mathcal{R}$ with $E(\varepsilon_{x,t}) = 0$ and $\text{Prob}(\varepsilon_{x,t} = 0) < 1$. Any outcome x to be received at $t = 0$ is perceived to be riskless, i.e. we assume that $\varepsilon_{x,0} = 0$ for all $x \in \mathbb{R}$. *Outcome risk for* x is *i.i.d.* if $\varepsilon_{x,t}$ is independently and identically distributed for all $t > 0$. *Outcome risk for x is increasing* if $\varepsilon_{x,\bar{t}}$ is a mean-preserving spread of $\varepsilon_{x,t}$ for all $0 \leq t < \bar{t} \leq T$.

In order to focus on the impact of outcome risk on behavior, throughout this section we will assume that endowments are stationary and riskless, i.e. $\omega = (w, \ldots, w)$ for some $w \in \mathbb{R}$. We start by defining an i- and d-reversal, which we will now call an outcome i- and outcome d-reversal, respectively. An outcome d-reversal suggests decreasing impatience and is defined as follows.

Definition 4.1 ***outcome d-reversal***

Induced preferences $\succcurlyeq^*$ exhibit an outcome d-reversal at the outcomes $y > x > 0$ and dates $\bar{t} > t \geq 0$ and $\tau > 0$ if

$$(t : x + \varepsilon_{x,t}) \sim^* (t + \tau : y + \varepsilon_{y,t+\tau}) \;\;\&\;\; (\bar{t} : x + \varepsilon_{x,\bar{t}}) \prec^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau}).$$

By contrast, an outcome i-reversal suggests increasing impatience.

Definition 4.2 ***outcome i-reversal***

Induced preferences $\succcurlyeq^*$ exhibit an outcome i-reversal at the outcomes $y > x > 0$ and dates $\bar{t} > t \geq 0$ and $\tau > 0$ if

$$(t : x + \varepsilon_{x,t}) \sim^* (t + \tau : y + \varepsilon_{y,t+\tau}) \;\;\&\;\; (\bar{t} : x + \varepsilon_{x,\bar{t}}) \succ^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau}).$$

Throughout this section we assume that the expected utility of receiving a risky outcome on top of the endowment is finite.

Assumption II $|EU(w + x + \varepsilon_{x,t})| < \infty$ for all $x \in \mathbb{R}$ and all $t = 0, \ldots, T$.

In Section 3 we showed that i-reversals at $t = 0$ are supported by a discount function that satisfies increasing, constant, or even decreasing impatience if endowment risk is i.i.d. Similarly, the next theorem shows that outcome d-reversals at $t = 0$ are supported by a discount function that satisfies decreasing, constant, or even increasing impatience if outcome risk is i.i.d. The intuition behind the result is as follows. Receiving a reward in the future instead of receiving it immediately has two drawbacks. First, under impatience a DM always prefers to receive a reward earlier. Second, in the future the reward will be subject to risk and a risk averse DM dislikes risk. Thus, for the DM to be indifferent between receiving reward x today and receiving reward y at the future date τ, reward y must be sufficiently superior to reward x so that it offsets the two aforementioned drawbacks. Now assume that outcome risk is i.i.d. Delaying the receipt of both outcomes by $\bar{t}$ periods does not increase the risk of outcome y, but does increase the risk of outcome x. Thus, this delay makes outcome x relatively less attractive, which results in a strict preference for the later outcome y.

Theorem 4.3 *Consider a DM with riskless endowment and i.i.d. outcome-risk for all $x \in \mathbb{R}$. For all outcomes $x \in \mathbb{R}$ let $\varepsilon_x \in \mathcal{R}$ be such that $\varepsilon_{x,t} = \varepsilon_x$ for all $t \geq 1$. Let $\succsim^*$ exhibit an outcome d-reversal at date 0, i.e.*

$$(0 : x) \sim^* (\tau : y + \varepsilon_y) \ \& \ (\bar{t} : x + \varepsilon_x) \prec^* (\bar{t} + \tau : y + \varepsilon_y),$$

where $y > x > 0$ and $\bar{t}, \tau > 0$, suggesting decreasing impatience. This reversal can be supported by a discount function satisfying decreasing, constant, or increasing impatience; in particular it can be supported by the discount function $\delta(t) = \exp(-bt^{1+g})$ for some $b, g > 0$.

Thus, if a DM (a) assumes that his endowment will not change in the future, and (b) thinks that outcome risk is i.i.d., then he will tend to exhibit d-reversals at $t = 0$, except when he has considerable increasing impatience.

The following theorem makes behavioral predictions. If a DM is indifferent between receiving outcome x today or outcome y at the future date τ, then decreasing or constant impatience implies that delaying the receipt of both outcomes by t periods results in a strict preference for the later outcome y. If the DM has a discount function $\delta(t) = \exp(-bt^{1+g})$ with $b, g > 0$ (i.e. increasing impatience), then there will be an outcome d-reversal for a given delay t whenever g is sufficiently small. Finally, if $\delta(t) = \exp(-bt^{1+g})$ and if there is an outcome d-reversal for some delay $\bar{t}$, then any delay $t < \bar{t}$ also results in a strict preference for the later outcome. Thus, if we observe one outcome d-reversal and if the assumptions in the next theorem are satisfied, then we will observe many more outcome d-reversals.

Theorem 4.4 *Consider a DM with riskless endowments and i.i.d. outcome-risk for all* $x \in \mathbb{R}$. *For all outcomes* $x \in \mathbb{R}$ *let* $\varepsilon_x \in \mathcal{R}$ *be such that* $\varepsilon_{x,t} = \varepsilon_x$ *for all* $t \geq 1$. *Let* $(0 : x) \sim^* (\tau : y + \varepsilon_y)$ *where* $y > x > 0$ *and* $\tau > 0$.

(i) *If the DM's discount function satisfies decreasing or constant impatience, then* $(t : x + \varepsilon_x) \prec^* (t + \tau : y + \varepsilon_y)$ *for all* $t > 0$.

(ii) *For every* $t > 0$ *there is a* G *such that if the DM has a discount function* $\delta(t) = \exp(-bt^{1+g})$ *with* $b, g > 0$ *and* $g < G$ *then* $(t : x + \varepsilon_x) \prec^* (t + \tau : y + \varepsilon_y)$

(iii) *If the DM's discount function is given by* $\delta(t) = \exp(-bt^{1+g})$ *for some* $b, g > 0$, *and if* $(\bar{t} : x + \varepsilon_x) \prec^* (\bar{t} + \tau : y + \varepsilon_y)$, *then* $(t : x + \varepsilon_x) \prec^* (t + \tau : y + \varepsilon_y)$ *for all* $0 < t \leq \bar{t}$.

As long as outcome-risk increases relatively strongly in the near future compared to the far future, an outcome d-reversal does not rule out increasing impatience, even when outcome risk is not i.i.d as is shown next.

Theorem 4.5 *Let induced preferences exhibit an outcome d-reversal, i.e.*

$$(t : x + \varepsilon_{x,t}) \sim^* (t + \tau : y + \varepsilon_{y,t+\tau}) \ \& \ (\bar{t} : x + \varepsilon_{x,\bar{t}}) \prec^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau})$$

where $y > x > 0$, $\bar{t} > t \geq 0$, *and* $\tau > 0$, *suggesting decreasing impatience. If outcome risk for* x *is increasing, then there exists increasing risk for* y, $\varepsilon_{y,t'}$, $t' = 1, \ldots, T$, *such that the outcome d-reversal can be supported by a discount function satisfying decreasing, constant, or increasing impatience; in particular it can be supported by a discount function* $\delta(t) = \exp(-bt^{1+g})$ *for some parameters* $b, g > 0$.

In the previous theorem we showed that there exist outcome risks $\varepsilon_{y,t'}$, $t' = 1, \ldots, T$, such that the outcome d-reversal can be supported by a discount function satisfying decreasing, constant, or increasing impatience. In the proof we constructed an increasing outcome risk for y. Alternatively, we could have let outcome y be riskless.

The next theorem shows that if outcome risk increases strongly in the near future and relatively weakly in the far future, then outcome d-reversals occur whenever the discount function has decreasing, constant, or moderate increasing impatience.

Theorem 4.6 *Let* $y > x > 0$, $\bar{t} > t \geq 0$, *and* $\tau > 0$. *Let outcome risk for* x *and* y *be increasing. Then, for every* M *there is an* N *such that if*

$$\begin{aligned} |EU(w + x + \varepsilon_{x,t}) - EU(w + x + \varepsilon_{x,\bar{t}})| &> M \text{ and} \\ |EU(w + y + \varepsilon_{y,t+\tau}) - EU(w + y + \varepsilon_{y,\bar{t}+\tau})| &< N. \end{aligned}$$

and if

$$(t : x + \varepsilon_{x,t}) \sim^* (t + \tau : y + \varepsilon_{y,t+\tau})$$

then the following holds.

(i) If the DM's discount function has decreasing or constant impatience then

$$(\bar{t} : x + \varepsilon_{x,\bar{t}}) \prec^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau}).$$

(ii) There is a G such that if the DM's discount function is $\delta(t) = \exp(-bt^{1+g})$ with $b, g > 0$ and $g < G$ then

$$(\bar{t} : x + \varepsilon_{x,\bar{t}}) \prec^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau}).$$

As long as outcome risk increases relatively strongly in the far compared to the near future, an outcome i-reversal does not rule out constant or decreasing impatience as is shown next.

Theorem 4.7 *Let induced preferences exhibit an outcome i-reversal, i.e.*

$$(t : x + \varepsilon_{x,t}) \sim^* (t + \tau : y + \varepsilon_{y,t+\tau}) \;\&\; (\bar{t} : x + \varepsilon_{x,\bar{t}}) \succ^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau})$$

where $y > x > 0$, $\bar{t} > t \geq 0$, and $\tau > 0$, suggesting increasing impatience. If outcome risk for y is increasing, then there exist $\varepsilon_{x,t'}$, $t' = 1, \ldots, T$, such that the outcome i-reversal can be supported by a discount function satisfying increasing, constant, or decreasing impatience; in particular it can be supported by a generalized hyperbolic discount function $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$ with $\alpha, \beta > 0$.

Finally, Theorem 4.8 shows that if outcome risk increases relatively strongly in the far future compared to the near future, then outcome i-reversals occur whenever the discount function satisfies increasing, constant, or moderate decreasing impatience.

Theorem 4.8 *Let $y > x > 0$, $\bar{t} > t \geq 0$, and $\tau > 0$. Let outcome risk for x and y be increasing. Then, for every M there is an N such that if*

$$\begin{aligned} |EU(w + y + \varepsilon_{y,t+\tau}) - EU(w + y + \varepsilon_{y,\bar{t}+\tau})| &> M \text{ and} \\ |EU(w + x + \varepsilon_{x,t}) - EU(w + x + \varepsilon_{x,\bar{t}})| &< N \end{aligned}$$

and if

$$(t : x + \varepsilon_{x,t}) \sim^* (t + \tau : y + \varepsilon_{y,t+\tau})$$

then the following holds.

(i) If the DM's discount function satisfies increasing or constant impatience then

$$(\bar{t} : x + \varepsilon_{x,\bar{t}}) \succ^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau}).$$

(ii) There is an A such that if the DM has a generalized hyperbolic discount function $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$ with $\alpha, \beta > 0$ and $\alpha < A$ then

$$(\bar{t} : x + \varepsilon_{x,\bar{t}}) \succ^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau}).$$

5 Magnitude effect and gain-loss asymmetry

In the previous sections we showed that perceived endowment and outcome risk can reconcile the empirical evidence in favor of decreasing impatience with Samuelson's constant discounting. For the sake of completeness we will argue now that two other anomalies in intertemporal choice can also be reconciled with the traditional constant discounted expected utility model: the magnitude effect and the gain-loss asymmetry. As we will show, these effects can be qualified as anomalies only under the assumption that subjects are risk neutral, i.e. have linear utility, an assumption that is frequently made in experimental studies.

There is evidence that large outcomes are discounted at lower rates than smaller ones, which is called the magnitude effect (Frederick, Loewenstein, and O'Donoghue, 2002; Kirby and Maraković, 1996; Loewenstein and Prelec, 1992; Thaler, 1981). When assuming linear utility this magnitude effect suggests that the discount factor depends not only on the date of receipt of an outcome, but also on the size of the outcome. Thus, the magnitude effect suggests that the discount function is increasing in the size of the outcome. We show here that the magnitude effect can be explained by concave utility only, even if endowment and outcomes are riskless, which we assume throughout this section. The magnitude effect is formally defined as follows.

Definition 5.1 ***magnitude effect***

Induced preferences $\succcurlyeq^*$ exhibit the magnitude effect at date t, and given outcomes x, y, X, Y with $Y > y > 0$, if

$$(0 : x) \sim^* (t : y) \;\; \& \;\; (0 : X) \sim^* (t : Y),$$

and $X/Y > x/y$.

The following example shows that the magnitude effect can occur with concave utility and all types of discounting. It is straightforward to see that similar examples can still be constructed if endowments or outcomes are risky.

Example 5.2 Let U be given by

$$U(x) = \begin{cases} x^{\frac{35}{48}} & , \text{ for } 0 \leq x \leq 1 \\ \frac{1}{4} + \frac{37}{48}x - \frac{1}{48}x^2 & , \text{ for } x > 1 \end{cases}$$

Then U is continuously differentiable and strictly concave. Let $w = 0.01, x = 0.99, y = 3.99, Y = 4.99$ and let X solve

$$\frac{U(w+x) - U(w)}{U(w+y) - U(w)} = \frac{U(w+X) - U(w)}{U(w+Y) - U(w)}.$$

Then $X \approx 1.25$ and hence

$$\frac{X}{Y} > \frac{x}{y}.$$

If we let $\delta(t) = \frac{U(w+x)-U(w)}{U(w+y)-U(w)}$, then the DM exhibits the magnitude effect. □

Thus, the magnitude effect does not necessarily imply that the discount function must be outcome dependent.

The gain-loss asymmetry states that losses are discounted at a lower rate than gains (Loewenstein and Prelec, 1992).

Definition 5.3 ***gain-loss asymmetry***

Induced preferences $\succcurlyeq^*$ exhibit the gain-loss asymmetry at date t, and outcomes x, y, X, Y with $Y, y > 0$, if

$$(0 : x) \sim^* (t : y) \;\; \& \;\; (0 : -X) \sim^* (t : -Y),$$

and $X/Y > x/y$.

As shown by Loewenstein and Prelec (1992, p. 576) the gain-loss asymmetry always occurs under constant discounted utility whenever the utility function is concave. It can easily be checked that the same holds for any kind of discount function. Thus, the gain-loss asymmetry does not necessarily imply that the discount functions for losses and for gains must be different.

6 Discussion

We have shown that non-stationary behavior can be observed whenever decision-makers perceive outcomes or endowments as risky. Though we considered endowment risk and outcome-risk in isolation, similar results would obtain if both risks are perceived simultaneously. In this case both risks interact and the dominant change in risk over time determines the observed behavior. Thus, the result that decreasing and increasing impatience can be observed simultaneously, even for a DM with Samuelson's constant discount function, still holds with this richer structure in risk.

We have also shown that the magnitude effect and the gain-loss asymmetry can be explained by concave utility whenever endowments and outcomes are riskless. Obviously, allowing for risk in endowments and/or outcomes, the two apparent anomalies can still be obtained under discounted expected utility if the utility function is concave.

7 Appendix

Proof of Theorems 3.2 and 3.3

The i-reversal can be supported by a discount function δ if

$$U(\omega_0 + x) - U(\omega_0) = \delta(\tau)[EU(\bar{\omega} + y) - EU(\bar{\omega})] \tag{2}$$

and

$$\delta(\bar{t})[EU(\bar{\omega} + x) - EU(\bar{\omega})] > \delta(\bar{t} + \tau)[EU(\bar{\omega} + y) - EU(\bar{\omega})]. \tag{3}$$

From $U(\omega_0) > EU(\bar{\omega})$ and the strict concavity of U it follows that

$$U(\omega_0) - U(E(\bar{\omega}) - \pi(0, \bar{\omega})) > U(\omega_0 + x) - U(x + E(\bar{\omega}) - \pi(0, \bar{\omega})).$$

Strictly decreasing absolute risk aversion implies that $\pi(x, \bar{\omega}) < \pi(0, \bar{\omega})$. Thus,

$$U(\omega_0) - U(E(\bar{\omega}) - \pi(0, \bar{\omega})) > U(\omega_0 + x) - U(x + E(\bar{\omega}) - \pi(x, \bar{\omega})).$$

It follows that

$$EU(\bar{\omega} + x) - EU(\bar{\omega}) > U(\omega_0 + x) - U(\omega_0).$$

Thus, there exists $\delta(\tau) < 1$ such that (2) holds. Then (3) is satisfied if and only if

$$\frac{\delta(\tau)\delta(\bar{t})}{\delta(\bar{t} + \tau)} [EU(\bar{\omega} + x) - EU(\bar{\omega})] > U(\omega_0 + x) - U(\omega_0), \tag{4}$$

which holds for any discount function δ that satisfies increasing or constant impatience. Thus, we have shown that the i-reversal can be supported by any discount function satisfying increasing or constant impatience and (2).

For decreasing impatience consider the generalized hyperbolic discount function $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$ with $\alpha, \beta > 0$. If we let

$$\beta = \beta(\alpha) := -\alpha A(\alpha),$$

where

$$A(\alpha) := \ln\left(\frac{U(\omega_0 + x) - U(\omega_0)}{EU(\bar{\omega} + y) - EU(\bar{\omega})}\right) / \ln(1 + \alpha\tau).$$

[4]The generalized hyperbolic discount function was proposed by Loewenstein and Prelec (1992).

then (2) is satisfied for any $\alpha > 0$. It is straightforward to show that

$$\lim_{\alpha\to 0}\frac{\delta(\tau)\delta(\bar{t})}{\delta(\bar{t}+\tau)} = \lim_{\alpha\to 0}\left[\frac{(1+\alpha\tau)(1+\alpha\bar{t})}{1+\alpha(\bar{t}+\tau)}\right]^{A(\alpha)} = 1.$$

Hence, for α sufficiently small both (2) and (4) are satisfied and we have shown that the i-reversal can be supported by a discount function satisfying decreasing impatience. This proves Theorem 3.2.

The proof of (i) and (ii) in Theorem 3.3 follows from the previous reasoning.

We will now prove (iii) of Theorem 3.3. Assume that $(0 : x) \sim^* (\tau : y)$ and $(\bar{t} : x) \succ^* (\bar{t}+\tau : y)$. From the proof of Theorem 3.2 we know that it must be the case that

$$1 > \frac{\delta(\tau)\delta(\bar{t})}{\delta(\bar{t}+\tau)} > \frac{U(\omega_0+x)-U(\omega_0)}{EU(\bar{\omega}+x)-EU(\bar{\omega})}.$$

Consider the function

$$f(t) = \frac{\delta(\tau)\delta(t)}{\delta(t+\tau)} = \frac{(1+\alpha\tau)^{-\beta/\alpha}(1+\alpha t)^{-\beta/\alpha}}{(1+\alpha(t+\tau))^{-\beta/\alpha}}.$$

It can easily be checked that $f(t)$ is strictly decreasing in $t > 0$. Thus,

$$1 > \frac{\delta(\tau)\delta(t)}{\delta(t+\tau)} > \frac{U(\omega_0+x)-U(\omega_0)}{EU(\bar{\omega}+x)-EU(\bar{\omega})} \tag{5}$$

for all $t \leq \bar{t}$. By a similar reasoning as in the proof of Theorem 3.2 it follows that $(t : x) \succ^* (t+\tau : y)$ for all t with $0 < t \leq \bar{t}$. This proves Theorem 3.3. □

Proof of Theorems 3.4 and 3.5

Consider endowments with increasing endowment-risk that satisfy assumption I. We start by making an observation which will be helpful in the proof of the result.

Since $U''' > 0$ by strictly decreasing absolute risk aversion and since $\omega_{\bar{t}}$ is a mean-preserving spread of ω_t it follows that

$$E\left[\frac{d}{dz}U(\omega_{\bar{t}}+z)\right] > E\left[\frac{d}{dz}U(\omega_t+z)\right].$$

Thus,

$$\frac{d}{dz}EU(\omega_{\bar{t}} + z) > \frac{d}{dz}EU(\omega_t + z)$$

for all $z \in \mathbb{R}$.

Thus,

$$EU(\omega_t + x) - EU(\omega_t) < EU(\omega_{\bar{t}} + x) - EU(\omega_{\bar{t}}). \tag{6}$$

We can now start with the proof of our result. From $(t : x) \sim^* (t + \tau : y)$ we know that for the reversal to be supported by a discount function δ we need

$$\delta(t)\left[EU(\omega_t + x) - EU(\omega_t)\right] = \delta(t+\tau)\left[EU(\omega_{t+\tau} + y) - EU(\omega_{t+\tau})\right]. \tag{7}$$

Consider the inequality

$$\frac{\delta(t+\tau)\delta(\bar{t})}{\delta(t)\delta(\bar{t}+\tau)}\left[EU(\omega_{\bar{t}} + x) - EU(\omega_{\bar{t}})\right] > EU(\omega_t + x) - EU(\omega_t). \tag{8}$$

Note first that for increasing and constant impatience the foregoing inequality always holds by (6). For decreasing impatience consider the generalized hyperbolic discount function $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$ with $\alpha, \beta > 0$. By a similar argument as in the proof of Theorem 3.2 there exist $\alpha > 0$ and $\beta > 0$ such that (7) and (8) are both satisfied.

Now we see that if

$$EU(\omega_{t+\tau} + y) - EU(\omega_{t+\tau})$$

is sufficiently close to

$$EU(\omega_{\bar{t}+\tau} + y) - EU(\omega_{\bar{t}+\tau})$$

then from (7) and (8) it follows that

$$\delta(\bar{t})\left[EU(\omega_{\bar{t}} + x) - EU(\omega_{\bar{t}})\right] > \delta(\bar{t}+\tau)\left[EU(\omega_{\bar{t}+\tau} + y) - EU(\omega_{\bar{t}+\tau})\right].$$

Hence, the i-reversal

$$(t : x) \sim^* (t + \tau, y) \;\&\; (\bar{t} : x) \succ^* (\bar{t} + \tau : y),$$

can be supported by a discount function satisfying increasing, constant or decreasing impatience.

It remains to be shown that there exist endowments with increasing endowment risk such that $EU(\omega_{t+\tau}+y)-EU(\omega_{t+\tau})$ is close to $EU(\omega_{\bar{t}+\tau}+y)-EU(\omega_{\bar{t}+\tau})$. To see this let $\gamma>0$ and let $\varepsilon_i\in\mathcal{R}$ be uniformly and independently distributed on $[-\gamma,\gamma]$ for all $i=1,\ldots,T-t-\tau$. Let $\omega_{t+\tau+i}=\omega_{t+\tau}+\sum_{k=1}^{i}\varepsilon_k$ for $i=1,\ldots,T-t-\tau$. Then $\omega_{t+\tau+j}$ is a mean-preserving spread of $\omega_{t+\tau+i}$ for all $j>i$. By continuity of U it follows that $EU(\omega_{\bar{t}+\tau}+y)-EU(\omega_{\bar{t}+\tau})$ converges to $EU(\omega_{t+\tau}+y)-EU(\omega_{t+\tau})$ if γ goes to 0 while the inequality in (8) still holds in the limit. This proves Theorem 3.4.

To prove Theorem 3.5 note that from $(t:x)\sim^*(t+\tau:y)$ and $|\,[EU(\omega_{\bar{t}}+x)-EU(\omega_{\bar{t}})]-[EU(\omega_t+x)-EU(\omega_t)]\,|>M$ it follows that (7) and (8) hold whenever δ satisfies increasing or constant impatience or whenever δ is of the generalized hyperbolic form, i.e. $\delta(t)=(1+\alpha t)^{-\beta/\alpha}$ with $\beta>0$ and $\alpha>0$ sufficiently small (cf. the proof of Theorem 3.2). Hence, by the same argument as in the first part of the proof, if for given M, $|\,[EU(\omega_{\bar{t}+\tau}+y)-EU(\omega_{\bar{t}+\tau})]-[EU(\omega_{t+\tau}+y)-EU(\omega_{t+\tau})]\,|$ is sufficiently small, then $(\bar{t}:x)\succ^*(\bar{t}+\tau:y)$. This proves Theorem 3.5. □

Proof of Theorems 3.7 and 3.8

Consider endowments with increasing endowment-risk that satisfy assumption I. By $(t:x)\sim^*(t+\tau:y)$ we know that for the reversal to be supported by a discount function δ we need

$$\delta(t)\left[EU(\omega_t+x)-EU(\omega_t)\right]=\delta(t+\tau)\left[EU(\omega_{t+\tau}+y)-EU(\omega_{t+\tau})\right]. \qquad (9)$$

By the same argument as in the proof of Theorem 3.4 we know that

$$EU(\omega_{\bar{t}+\tau}+y)-EU(\omega_{\bar{t}+\tau})>EU(\omega_{t+\tau}+y)-EU(\omega_{t+\tau}).$$

It follows that whenever δ satisfies constant or decreasing impatience we have

$$EU(\omega_{\bar{t}+\tau}+y)-EU(\omega_{\bar{t}+\tau})>\frac{\delta(t+\tau)\delta(\bar{t})}{\delta(t)\delta(\bar{t}+\tau)}\left[EU(\omega_{t+\tau}+y)-EU(\omega_{t+\tau})\right]. \qquad (10)$$

For increasing impatience consider the discount function $\delta(t) = \exp(-bt^{1+g})$ for $b, g > 0$. Define

$$b = b(g) := \frac{\ln(A)}{(t+\tau)^{1+g} - t^{1+g}},$$

where

$$A := \frac{EU(\omega_{t+\tau} + y) - EU(\omega_{t+\tau})}{EU(\omega_t + x) - EU(\omega_t)}.$$

Then (9) is satisfied for all $g > 0$. It is immediate to see that $\delta(t') \to A^{-t'/\tau}$ for all t' if $g \to 0$. Hence,

$$\lim_{g \to 0} \frac{\delta(t+\tau)\delta(\bar{t})}{\delta(t)\delta(\bar{t}+\tau)} = 1$$

Thus, for g sufficiently small both (9) and (10) are satisfied, which implies

$$EU(\omega_{\bar{t}+\tau} + y) - EU(\omega_{\bar{t}+\tau}) > \frac{\delta(\bar{t})}{\delta(\bar{t}+\tau)} \left[EU(\omega_t + x) - EU(\omega_t) \right].$$

It remains to show that there exist endowments with increasing endowment risk such that $EU(\omega_t + x) - EU(\omega_t)$ is sufficiently close to $EU(\omega_{\bar{t}} + x) - EU(\omega_{\bar{t}})$ so that

$$EU(\omega_{\bar{t}+\tau} + y) - EU(\omega_{\bar{t}+\tau}) > \frac{\delta(\bar{t})}{\delta(\bar{t}+\tau)} \left[EU(\omega_{\bar{t}} + x) - EU(\omega_{\bar{t}}) \right].$$

To see this let $\gamma, \lambda > 0$, and let $\varepsilon_t \in \mathcal{R}$ be uniformly and independently distributed on $[-\gamma, \gamma]$ for all $t = 1, \ldots, \bar{t}$, and let $\varepsilon_t \in \mathcal{R}$ be uniformly distributed on $[-\lambda, \lambda]$ for all $t = \bar{t}+1, \ldots, T$. Let $\omega_t = \omega_0 + \sum_{k=1}^{t} \varepsilon_k$ for $t = 1, \ldots, T$. Then $\omega_{t'}$ is a mean-preserving spread of ω_t for all $t' > t$. By continuity of U it follows that $EU(\omega_{\bar{t}} + x) - EU(\omega_{\bar{t}})$ converges to $EU(\omega_t + x) - EU(\omega_t)$ if γ goes to 0, while the inequality in (10) still holds in the limit. It follows that $(\bar{t} : x) \prec^* (\bar{t}+\tau : y)$. This proves Theorem 3.7.

To prove Theorem 3.8 note that from $(t : x) \sim^* (t+\tau : y)$ and $|\,[EU(\omega_{\bar{t}+\tau} + y) - EU(\omega_{\bar{t}+\tau})] - [EU(\omega_{t+\tau} + y) - EU(\omega_{t+\tau})]\,| > M$ it follows that (9) and (10) hold whenever δ satisfies decreasing or constant impatience or whenever $\delta(t) = \exp(-bt^{1+g})$ for $b > 0$ and $g > 0$ sufficiently small. Hence, by the same argument as in the first part of the proof, if for given M, $|\,[EU(\omega_{\bar{t}} + x) - EU(\omega_{\bar{t}})] - [EU(\omega_t + x) - EU(\omega_t)]\,|$ is sufficiently small, then $(\bar{t} : x) \prec^* (\bar{t}+\tau : y)$. This proves Theorem 3.8. □

Proof of Theorems 4.3 and 4.4

Induced preferences exhibit the outcome d-reversal $(0 : x) \sim^* (\tau : y+\varepsilon_y)$ and $(\bar{t} : x+\varepsilon_x) \prec^* (\bar{t}+\tau : y+\varepsilon_y)$ if and only if the discount function satisfies

$$\begin{aligned} U(w+x) - U(w) &= \delta(\tau)\left[EU(w+y+\varepsilon_y) - U(w)\right], \\ \text{and} \quad \delta(\bar{t}+\tau)[EU(w+y+\varepsilon_y) - U(w)] &> \delta(\bar{t})[EU(w+x+\varepsilon_x) - U(w)], \end{aligned}$$

which is equivalent to

$$U(w+x) - U(w) = \delta(\tau)\left[EU(w+y+\varepsilon_y) - U(w)\right], \tag{11}$$

$$\text{and} \quad U(w+x) - U(w) > \frac{\delta(\tau)\delta(\bar{t})}{\delta(\bar{t}+\tau)}\left[EU(w+x+\varepsilon_x) - U(w)\right]. \tag{12}$$

By strict concavity of U we have $U(w+x) > EU(w+x+\varepsilon_x)$. Hence, the outcome d-reversal is supported by any discount function that satisfies decreasing or constant impatience and for which $\delta(\tau)$ satisfies (11). It is also supported by any discount function that satisfies increasing impatience as long as the increase in impatience is sufficiently moderate, so that (12) holds. For a particular class of discount functions satisfying increasing impatience consider the discount function $\delta(t) = \exp(-bt^{1+g})$ for $b, g > 0$. By a similar argument as in the proof of Theorem 3.7 one can show that there exist parameters b and g with g sufficiently close to zero such that δ satisfies (11) and (12). We have now proven Theorem 4.3.

The proof of items (i) and (ii) of Theorem 4.4 follows from the above reasoning. We will now prove (iii) of Theorem 4.4. Assume that $(0 : x) \sim^* (\tau : y+\varepsilon_y)$ and $(\bar{t} : x+\varepsilon_x) \prec^* (\bar{t}+\tau : y+\varepsilon_y)$. Then

$$U(w+x) - U(w) > \frac{\delta(\tau)\delta(\bar{t})}{\delta(\bar{t}+\tau)}\left[EU(w+x+\varepsilon_x) - U(w)\right].$$

Consider the function

$$f(t) = \frac{\delta(\tau)\delta(t)}{\delta(t+\tau)} = \exp\left[-b\left(\tau^{1+g} + t^{1+g} - (t+\tau)^{1+g}\right)\right].$$

It can easily be checked that $f(t)$ is strictly increasing in t. Thus,

$$U(w+x) - U(w) > \frac{\delta(\tau)\delta(t)}{\delta(t+\tau)}\left[EU(w+x+\varepsilon_x) - U(w)\right].$$

for all $t \leq \bar{t}$. Hence, $(t : x + \varepsilon_x) \prec^* (t + \tau : y + \varepsilon_y)$ for all t with $0 < t \leq \bar{t}$. $\square$

Proof of Theorems 4.5 and 4.6

We start with the proof of Theorem 4.5. From $(t : x + \varepsilon_{x,t}) \sim^* (t + \tau : y + \varepsilon_{y,t+\tau})$ it follows that the discount function δ must satisfy

$$\delta(t)\left[EU(w + x + \varepsilon_{x,t}) - U(w)\right] = \delta(t+\tau)\left[EU(w + y + \varepsilon_{y,t+\tau}) - U(w)\right]. \qquad (13)$$

If outcome risk for x is increasing then

$$EU(w + x + \varepsilon_{x,t}) - U(w) > EU(w + x + \varepsilon_{x,\bar{t}}) - U(w).$$

Hence, if δ satisfies decreasing or constant impatience we have

$$EU(w + x + \varepsilon_{x,t}) - U(w) > \frac{\delta(t+\tau)\delta(\bar{t})}{\delta(t)\delta(\bar{t}+\tau)} EU(w + x + \varepsilon_{x,\bar{t}}) - U(w). \qquad (14)$$

By a similar reasoning as in the proof of Theorem 3.7, there exist $b, g > 0$ with g close to zero such that (13) and (14) also hold for $\delta(t) = \exp(-bt^{1+g})$, i.e. for a discount function that satisfies increasing impatience. From (13) and (14) it follows that

$$EU(w + y + \varepsilon_{y,t+\tau}) - U(w) > \frac{\delta(\bar{t})}{\delta(\bar{t}+\tau)} EU(w + x + \varepsilon_{x,\bar{t}}) - U(w).$$

Hence, if $EU(w + y + \varepsilon_{y,\bar{t}+\tau})$ is close to $EU(w + y + \varepsilon_{y,t+\tau})$, then

$$EU(w + y + \varepsilon_{y,\bar{t}+\tau}) - U(w) > \frac{\delta(\bar{t})}{\delta(\bar{t}+\tau)} EU(w + x + \varepsilon_{x,\bar{t}}) - U(w),$$

which implies $(\bar{t} : x + \varepsilon_{x,\bar{t}}) \prec^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau})$. It remains to show that there exist $\varepsilon_{y,t'}$, $t' = 1, \ldots, T$, such that $EU(w + y + \varepsilon_{y,\bar{t}+\tau})$ is close to $EU(w + y + \varepsilon_{y,t+\tau})$. Let $\gamma > 0$ and let $\varepsilon_{t'}$ be independently and uniformly distributed on $[-\gamma, \gamma]$ for all $t' = 1, \ldots, T$. Let $\varepsilon_{y,t'} = \varepsilon_1 + \cdots + \varepsilon_{t'}$ for all $t' = 1, \ldots, T$. By continuity of U it follows that $EU(w + y + \varepsilon_{y,\bar{t}+\tau})$ converges to $EU(w + y + \varepsilon_{y,t+\tau})$ if γ goes to 0. This proves Theorem 4.5.

To prove Theorem 4.6 note that from $(t : x + \varepsilon_{x,t}) \sim^* (t + \tau : y + \varepsilon_{y,t+\tau})$ and $|EU(w + x + \varepsilon_{x,t}) - EU(w + x + \varepsilon_{x,\bar{t}})| > M$ it follows that (13) and (14) hold whenever δ satisfies decreasing or constant impatience or whenever $\delta(t) = \exp(-bt^{1+g})$ for $b > 0$ and $g > 0$ sufficiently small. Hence, by the same argument as in the first part of the proof, if for given M, $|EU(w + y + \varepsilon_{y,t+\tau}) - EU(w + y + \varepsilon_{y,\bar{t}+\tau})|$ is sufficiently small, then $(\bar{t} : x + \varepsilon_{x,\bar{t}}) \prec^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau})$. This proves Theorem 4.6. □

Proof of Theorems 4.7 and 4.8

From $(t : x + \varepsilon_{x,t}) \sim^* (t + \tau : y + \varepsilon_{y,t+\tau})$ it follows that the discount function δ must satisfy

$$\delta(t)\left[EU(w + x + \varepsilon_{x,t}) - U(w)\right] = \delta(t + \tau)\left[EU(w + y + \varepsilon_{y,t+\tau}) - U(w)\right], \tag{15}$$

Since outcome risk for y is increasing we know that

$$EU(w + y + \varepsilon_{y,t+\tau}) - U(w) > EU(w + y + \varepsilon_{y,\bar{t}+\tau}) - U(w).$$

Hence, if δ satisfies increasing or constant impatience we have

$$\frac{\delta(t+\tau)\delta(\bar{t})}{\delta(t)\delta(\bar{t}+\tau)} EU(w + y + \varepsilon_{y,t+\tau}) - U(w) > EU(w + y + \varepsilon_{y,\bar{t}+\tau}) - U(w). \tag{16}$$

By a similar reasoning as in the proof of Theorem 3.2, there exist $\alpha, \beta > 0$ with α close to zero such that (15) and (16) also hold for $\delta(t) = (1 + \alpha t)^{-\beta/\alpha}$, i.e. for a discount function that satisfies decreasing impatience. From (15) and (16) it follows that

$$\frac{\delta(\bar{t})}{\delta(\bar{t}+\tau)} EU(w + x + \varepsilon_{x,t}) - U(w) > EU(w + y + \varepsilon_{y,\bar{t}+\tau}) - U(w).$$

Hence, if $EU(w + x + \varepsilon_{x,\bar{t}})$ is close to $EU(w + x + \varepsilon_{x,t})$, then

$$\frac{\delta(\bar{t})}{\delta(\bar{t}+\tau)} EU(w + x + \varepsilon_{x,\bar{t}}) - U(w) > EU(w + y + \varepsilon_{y,\bar{t}+\tau}) - U(w).$$

which implies $(\bar{t} : x + \varepsilon_{x,\bar{t}}) \succ^* (\bar{t} + \tau : y + \varepsilon_{y,\bar{t}+\tau})$. It remains to show that there exist $\varepsilon_{x,t'}$, $t' = 1, \ldots, T$, such that $EU(w + x + \varepsilon_{x,\bar{t}})$ is close to $EU(w + x + \varepsilon_{x,t})$. This follows from a similar argument as in the proof of Theorem 4.5. This proves Theorem 4.7. The proof of Theorem 4.8 is similar to the proof of Theorem 4.6. □

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