On the intensity of downside risk aversion

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Abstract

The degree of downside risk aversion (or equivalently prudence) is so far usually measured by $\frac{-U'''}{U''}$. We propose here another measure, $\frac{U'''}{U'}$, which has interesting properties, different from those related to $\frac{-U'''}{U''}$. It also appears that the two measures are not mutually exclusive. Instead, they seem to be rather complementary as shown through an economic application.

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1 Introduction

It is now common knowledge that in the expected utility model risk aversion is expressed by a negative second derivative of the utility function ($U'' < 0$) while its intensity is measured by $\frac{-U''}{U'}$, the degree of absolute risk aversion (Arrow, 1965; Pratt, 1964).

For downside risk aversion (henceforth D.R.A.), matters are not quite so obvious. There is unanimity to define D.R.A. in the expected utility model by a positive third derivative of the utility function ($U''' > 0$). However as far as its intensity is concerned, unanimity breaks down. It is true that most economists suggest to measure the intensity of downside risk aversion only by $\frac{-U'''}{U''}$.This choice is rationalized by the fact the downside risk aversion and prudence are the same thing (both are defined by $U''' > 0$)

while Kimball (1990) has convincingly argued in favor of $\frac{-U'''}{U''}$ as a measure of the prudence motive. While $\frac{-U'''}{U''}$ and its properties (e.g. "decreasing prudence") appeared to be very useful in many contexts so that $\frac{-U'''}{U''}$ was considered as *the* measure of downside risk aversion, two recent papers - to the least of our knowledge - questioned this point of view. The first one by Keenan and Snow (2002) is based on the notion of compensated increases in risk as defined by Diamond and Stiglitz (1974). In that context they obtain that the index of downside risk aversion is made of two terms, one of them being equal to $\frac{-U'''}{U''}$[1]. More recently, Modica and Scarsini (2006), starting from Ross' notion of stronger risk aversion (1981) obtained that the degree of local downside risk aversion should be measured by $\frac{U'''}{U'}$. However they observed thet "this coefficient does not allow to translate a local comparison into a global comparison" (see page 270 in their paper) contrarily to what happens for the Arrow Prattt coefficient of the absolute risk aversion.

The purpose of the present paper is twofold. First we want to derive $\frac{U'''}{U'}$ as a measure of D.R.A. intensity directly from an approach à la Arrow-Pratt[2]. We will show that this local measure does have interesting global properties and that it can give rise to relevant economic interpretations.

More specifically the paper is organized as follows. In section 2 the main result is presented and an intuitive approach is offered in section 3. The global properties of the local measure $\frac{U'''}{U'}$ are discussed in section 4 while an economic interpretation of the new measure of D.R.A. is given in section 5. In the conclusion we discuss the complementarity between the usual measure and the new one.

2 The main result

From Menezes, Geiss and Tressler (1980) and its presentation in Eeckhoudt and Schlesinger (2006) (from now on E-S) we know that in the expected utility model:

$$\frac{1}{2}U(x-k) + \frac{1}{2}EU(x+\widetilde{\epsilon}) > \frac{1}{2}U(x) + \frac{1}{2}EU(x-k+\widetilde{\epsilon}) \Longleftrightarrow U''' > 0 \quad (1)$$

where x is initial wealth, k is a positive constant and $\widetilde{\epsilon}$ a zero mean risk.

In E-S's terminology the left hand side (LHS) of (1) is preferred to the right hand side (RHS) because for the LHS term, the pains ($-k$ and $\widetilde{\epsilon}$) are

[1] The other is related to the degree of absolute risk aversion. See equation (7) and its discussion.

[2] Notice that the two papers just mentioned didn't use the Arrow-Pratt approach.

"better apportioned" than for the RHS term. More precisely, on the LHS, the pains are disaggregated while it is not the case on the RHS where they are concentrated on a single state of nature.

Since with $U''' > 0$ welfare is higher on the LHS of (1) we can then raise the following question (in the spirit of the definition of the risk premium): how much money (m) is the individual willing to give up in order to avoid the "misapportionment" on the RHS of (1)? Formally m is defined by:

$$\frac{1}{2}U(x-k-m)+\frac{1}{2}EU(x+\widetilde{\epsilon}-m)=\frac{1}{2}U(x)+\frac{1}{2}EU(x-k+\widetilde{\epsilon}) \qquad (2)$$

and of course

$$U''' \lesseqgtr 0 \Longleftrightarrow m \lesseqgtr 0$$

It can easily be shown that m is necessarily smaller than k.

In (2) m is thus the monetary value of the prudence motive because of the equivalence between the concept of risk apportionment and that of prudence (or downside risk aversion) as shown in E-S.

Besides, returning to each term in (1), we can also define amounts of money denoted respectively $P_1(x)$ and $P_2(x)$ such that all the risks involved in each term of (1) are eliminated. We thus have:

$$B \equiv \frac{1}{2}U(x-k)+\frac{1}{2}EU(x+\widetilde{\epsilon})=U(x-P_1(x)) \qquad (3')$$

$$A \equiv \frac{1}{2}U(x)+\frac{1}{2}EU(x-k+\widetilde{\epsilon})=U(x-P_2(x)) \qquad (3")$$

Clearly $P_1(x)$ and $P_2(x)$ correspond each to a risk premium à la Arrow-Pratt and again, under $U''' > 0$, $P_1(x) < P_2(x)$ since $B \succ A$.

Now it can be shown that:

$$m = P_2(x) - P_1(x-m) \qquad (4)$$

Indeed $P_2(x)$ is the willingness to pay to fully eliminate all the risk involved in A. To arrive at that result equation (4) says that one can proceed in two steps: first loose m to arrive at the risks involved in B and then pay $P_1(x)$ to fully eliminate these risks.

For small k and hence for small m since - as we know - $k < m$, (4) can be approximated by:

$$m \approx P_2(x) - P_1(x) \qquad (4')$$

If the zero mean risk $\widetilde{\epsilon}$ is small we can proceed as in Arrow-Pratt and write:

$$\frac{1}{2}U(x-k)+\frac{1}{2}\left[U(x)+\frac{\sigma^2}{2}U''(x)\right]\approx U(x)-P_1(x)U'(x) \qquad (5')$$

$$\frac{1}{2}U(x)+\frac{1}{2}\left[U(x-k)+\frac{\sigma^2}{2}U''(x-k)\right]\approx U(x)-P_2(x)U'(x) \qquad (5'')$$

where σ^2 is the variance of $\widetilde{\epsilon}$.

Substracting (5") from (5') yields:

$$\left(P_2(x)-P_1(x)\right)U'(x)=\frac{1}{2}\frac{\sigma^2}{2}\left(U''(x)-U''(x-k)\right)$$

For small k and using (4) we finally obtain:

$$m\approx\frac{\sigma^2}{4}k\frac{U'''(x)}{U'(x)} \qquad (6)$$

where $\frac{U'''(x)}{U'(x)}$ is the strength of the D.R.A. motive which confirms from another point of view the interest of the measure proposed by M-S.

Notice that - quite interestingly - m depends not only upon σ^2 as it is usual in the theory of risk aversion but also upon the value of k which is an indication of the importance of the misapportionment between A and B.

To illustrate the quality of the approximation proposed in (6) we solve in Appendix 1 a problem where an explicit solution for m can be obtained and we compare it with its approximation.

3 An intuitive approach

In a now much neglected paper, Friedman and Savage (1948) had defined the "utility premium" denoted R by:

$$R=U(x)-EU(x+\widetilde{\epsilon})$$

For concave U, R is positive and measures the pain inflicted by the presence of a zero mean risk. Using the standard approximation for small risks:

$$R\approx\frac{\sigma^2}{2}\left(-U''(x)\right)$$

As is well known, the risk premium (which is an amount of money, denoted π) is given by:

$$\pi \approx \frac{\sigma^2}{2}\left(\frac{-U''(x)}{U'(x)}\right)$$

or

$$\pi \approx \frac{R}{U'(x)}$$

The division of R by $U'(x)$ to generate π can then be easily understood: its purpose is to convert into money the pain due to risk $\widetilde{\epsilon}$ and the conversion factor is the marginal utility of money. The higher $U'(x)$, the lower the amount of money a decision maker is willing to give up in order to eliminate a given pain[3].

In fact a similar approach can be used to justify the use of $\frac{U'''}{U'}$ as a measure of the intensity of D.R.A. The pain due to a misapportionment of a risk $\widetilde{\epsilon}$ (denoted S) is defined by:

$$S = \frac{1}{2}U(x-k) + \frac{1}{2}EU(x+\widetilde{\epsilon}) - \frac{1}{2}U(x) - \frac{1}{2}EU(x-k+\widetilde{\epsilon})$$

For a small risk $\widetilde{\epsilon}$, one has:

$$S \approx \frac{1}{2}\frac{\sigma^2}{2}\left[U''(x) - U''(x-k)\right]$$

and then for small k:

$$S \approx \frac{\sigma^2}{4}kU'''(x) \tag{7}$$

To transform this pain into a money equivalent we have to divide by $U'(x)$ (as for the risk premium) and we then obtain:

$$m \approx \frac{\sigma^2}{4}k\frac{U'''(x)}{U'(x)}$$

as in (6).

This result not only confirms $\frac{U'''}{U'}$ as a measure of downside risk aversion. It also establishes the close link with the well accepted measure of absolute risk aversion ($\frac{-U''}{U'}$).

[3]Many more properties of the utility premium can be found in Eeckhoudt and Schlesinger (2006). Notice also that a similar approach is used by Foncel and Treich (2005) to analyse the concept of fear of ruin developed by Aumann and Kurz (1977) and which is equal to $\frac{U}{U'}$.

4 A global property

As is well known the Arrow-Pratt local index of absolute risk aversion has a nice global property, namely that if $V(x) = s(U(x))$ with $s' > 0$ and $s'' < 0$ then V has a higher degree of absolute risk aversion than U at any x. Hence it is natural to wonder if a local measure of the intensity of D.R.A. posesses a similar property.

For Kimball's local measure ($\frac{-U'''}{U''}$) it is already known that global properties are not easily obtained. As shown in Eeckhoudt and Schlesinger (1994)[4], if $s' > 0$ and $s'' < 0$ prevail so that V is more risk averse than U, the additional condition $s''' > 0$ implies more D.R.A. only if quite demanding restrictions are imposed on the utility function.

When $\frac{U'''}{U'}$ is used as the local measure of the intensity of D.R.A., matters are much easier.

Indeed let $V(x) = s(U(x))$ with $s' > 0$, $s'' < 0$ and $s''' > 0$. Then one easily obtains:

$$\frac{V'''}{V'} = \frac{s'''}{s'}\left(U'\right)^2 + 3\frac{s''}{s'}U'' + \frac{U'''}{U'}$$

Since $3\frac{s''}{s'}U''$ is positive for risk averse decision makers, the joint conditions $s'' < 0$ and $s''' > 0$ are sufficient[5] to imply that the utility function V simultaneously displays more risk aversion and more downside risk aversion than U at any x.

It thus appears that the local measure $\frac{U'''}{U'}$ has global properties that are much more intuitive than those attached to $\frac{-U'''}{U''}$.

5 An economic application

Following Kimball (1990) it is now well known that the local coefficient of prudence $\frac{-U'''}{U''}$ can be used to characterize precautionary savings motive. When $\frac{-U'''}{U''}$ increases *caeteris paribus* the decision maker increases the quantity of precautionary savings he wants to build up in order to better face future income risk. Hence a link is created between $\frac{-U'''}{U''}$ and a *quantity* demanded (in this case that of precautionary savings).

We now show that the other measure of the prudence motive ($\frac{U'''}{U'}$) is linked to an equilibrium *price* (instead of a quantity demanded). In order to maintain the link with the savings literature consider a very simple economy

[4] See especially their corollary 3.4.

[5] $s''' > 0$ is sufficient but not necessary to yield the result.

with no physical investment and where future consumption ($\widehat{c}_2$) is random because future income y_2 is risky. In this economy the representative consumer maximizes:

$$U(c_1) + \frac{1}{1+\theta} E\left[U((y_1 - c_1)(1+r) + \widehat{y}_2)\right] \tag{8}$$

where θ is the discount rate applied to the next period utility and r is the safe return on savings. As usual, c_1 and y_1 represent respectively current consumption and current income.

The optimality condition is written:

$$U'(c_1) + \frac{(1+r)}{1+\theta} E\left[U'((y_1 - c_1)(1+r) + \widehat{y}_2)\right] = 0 \tag{9}$$

If there is no investment, in equilibrium $c_1 = y_1$ and the optimality condition becomes then[6]:

$$U'(c_1) - \frac{(1+r)}{1+\theta} E\left[U'(\widehat{y}_2)\right] = 0 \tag{10}$$

When $\widehat{y}_2$ is non random and if $y_1 = y_2$ the equilibrium rate of return on saving is equal to the discount rate θ.

Should $\widehat{y}_2$ become random with $E(\widehat{y}_2) = y_1$ and if utility is quadratic, the uncertainty on $\widehat{y}_2$ has no effect on the equilibrium rate r essentially because with a quadratic utility $U''' = 0$. When U is quadratic future income uncertainty does not generate an additional savings demand so that the equilibrium rate of interest is not affected.

In order to isolate the effect of a D.R.A. on r, consider now that U is cubic with consumption levels such that $U' > 0$, $U'' < 0$ and $U''' > 0$[7]. In this case we have:

$$E\left[U'(\widehat{y}_2)\right] = U'(E(\widehat{y}_2)) + \frac{\sigma^2}{2} U'''(E(\widehat{y}_2))$$

so that for $E\left[U'(\widehat{y}_2)\right] = y_1$, (10) becomes:

$$U'(y_1) - \frac{(1+r)}{1+\theta} U'(y_1) \left[1 + \frac{\sigma^2}{2} \frac{U'''(y_1)}{U'(y_1)}\right] = 0 \tag{11}$$

[6] Condition (10) is one of the two equilibrium conditions discussed in Barsky (1989). Barsky's model is more general than the present one since it allows for the existence of a safe and a risky asset.

[7] By limiting ourselves to cubic utility we do not introduce effects of higher orders such as temperance or edginess that affect also the equilibrium value of r.

Hence in equilibrium one now has:

$$(1+r) = \frac{1+\theta}{1+\frac{\sigma^2}{2}\frac{U'''(y_1)}{U'(y_1)}} \tag{12}$$

The presence of future income uncertainty depresses the equilibrium rate of interest and it becomes lower than θ. This result is in accordance with intuition since the increased saving generated by future income risk has to remain equal in equilibrium to the fixed zero level of investment and this can be achieved only by a fall in r. What equation (12) reveals further is that the impact on equilibrium r of future income uncertainty (expressed by σ^2, the variance of $\widehat{y}_2$) is related to the size of $\frac{U'''}{U'}$, the alternative measure of D.R.A. intensity. When $\frac{U'''}{U'}$ increases, caeteris paribus, the equilibrium rate of return on savings falls.

This result clarifies the respective roles of the two "competing" measures of downside risk aversion: while $\frac{-U'''}{U''}$ is useful to interpret changes in (market) quantities, $\frac{U'''}{U'}$ is related to the induced change in market equilibrium price. This result should not be surprising since as shown in section 3 the division of the pain due to misapportionment by U' leads to a money equivalent which is indeed close to the notion of price.

6 Conclusion

The local measure of prudence (or downside risk aversion) suggested by Kimball, $\frac{-U'''}{U''}$, has been very useful to develop new tools for the economic analysis under risk.

In this paper we have proposed another local measure, $\frac{U'''}{U'}$, which turns out to also have interesting properties. First, it is related quite directly to the standard concept of absolute risk aversion, as shown in section 2 and 3. Besides it has a global property that is a rather natural extension of the one obtained for the coefficient of absolute risk aversion.

All this doesn't mean that one measure dominates the other. Instead they are more complementary than competitors. For instance, as indicated in section 5, $\frac{-U'''}{U''}$ explains changes in quantities demanded or supplied while $\frac{U'''}{U'}$ is more appropriate to discuss changes in equilibrium prices.

7 References

Aumann, Robert and Mordecai Kurz. (1977). "Power and taxes," *Econometrica* 45(5), 1137-1161.

Arrow, Kenneth (1965) "Aspects of the theory of risk-bearing", Helsinki, Yrjš Jahnsson Foundation.

Barsky, Robert (1989), "Why don't the prices of stocks and bonds move together?, *American Economic Review* 79, 1132-1145"

Diamond, Peter and Joseph Stiglitz. (1974). "Increases in risk and in risk aversion," *Journal of Economic Theory* 8(3), 335-360.

Eeckhoudt, Louis and Harris Schlesinger. (2006). "Putting risk in its proper place," *American Economic Review* 96(1), 280-289.

Eeckhoudt, Louis and Harris Schlesinger. (2005). "On the utility premium of Friedman and Savage," *mimeo*, 18 pages.

Eeckhoudt, Louis and Harris Schlesinger. (1994). "A precautionary tale of risk aversion and prudence," in *Models and experiments in risk and rationality*, B. Munier and M. Marchand, eds. Kluwer Academic Publishers, 75-90.

Foncel, Jérôme and Nicolas Treich. (2005). "Fear of ruin," *The Journal of Risk and Uncertainty* 31(3), 289-300.

Friedman, Milton, Leonard Savage. (1948). "The utility analysis of choices involving risk," *Journal of Political Economy* 56(4), 279-304.

Keenan Donald and Arthur Snow. (2005). "Greater downside risk aversion," *The Journal of Risk and Uncertainty* 24 (3), 267-277.

Kimball, Miles. (1990). "Precautionary saving in the small and in the large," *Econometrica* 58(1), 53-73.

Modica, Salvatore and Marco Scarsini. (2005). "A note on comparative downside risk aversion," *Journal of Economic Theory* 122, 267-271

Menezes Carmen, C. Geiss and John Tressler. (1980). "Increasing downside risk," *American Economic Review* 70(5), 921-932.

Pratt, John. (1964). "Risk aversion in the small and in the large," *Econometrica* 32, 122-136.

Ross, Stephen. (1981). "Some stronger measures of risk aversion in the small and the large with applications," *Econometrica* 49(3), 621-638.