

## QUASI RISK-NEUTRAL PRICING

BY

HARRY NIEDERAU AND PETER ZWEIFEL

## ABSTRACT

This contribution shows that for certain classes of insurance risks, pricing can be based on expected values under a unique equivalent probability measure  $\mathbb{P}^*$  amounting to quasi risk-neutral pricing. This probability measure is optimal in the sense of minimizing relative entropy with respect to the actuarial probability measure, which is a common approach in the case of incomplete markets. After expounding the key elements of this theory, an application to an industrial property book is developed, assuming that the severity of losses can be modeled by “Swiss Re Exposure Curves”, as discussed by Bernegger [1997]. These curves belong to a parametric family of distribution functions commonly used by pricing actuaries. The quasi risk-neutral pricing approach not only yields a set of location-specific insurance premiums but also Risk Adjusted Capital (RAC) values. By way of contrast, the conventional determination of RAC is typically limited entire portfolios.

## KEYWORDS

Esscher transform, incomplete markets, insurance prices, Lorenz order, quasi risk-neutral pricing, risk aversion, risk management, Risk Adjusted Capital (RAC).

## 1. INTRODUCTION

The theory of pricing risk in incomplete markets based on risk-neutral valuation is well established [see e.g. Goovaerts et al, 1984, Delbaen and Haezendonck, 1989, or Gerber and Shiu, 1994]. While a risk-neutral probability measure is not unique in an incomplete market, various approaches have been proposed and discussed for achieving “minimum distance” of the risk-neutral probability measure with respect to the physical probability measure [see e.g. Föllmer and Schweizer, 1991].

Risk measures used in practice, such as expected shortfall [see Artzner et al., 1999], relate to insurance portfolios rather than individual risks. As a consequence,

ad hoc pricing methods continue to be very much in use when it comes to pricing individual risks. This is surprising since distortion principles provide a consistent approach to pricing single risks in general, and tail risk in particular [see e.g. Wang, 1995, 1997, or Denneberg, 1990, for some earlier work on distorted probabilities]. With respect to assets, Wang [2000] has made an attempt to bridge the gap between theory and practice by showing that the Black-Scholes formula can be reproduced by a distortion principle relating to Choquet pricing [see e.g. de Waegenaere et al., 2003].

The idea of this work is to apply Choquet pricing to industrial property insurance, which by the infrequent nature of loss events (excess of some basic threshold) is typically exposed to heavy tails. Although distortion principles do take into account the extra loading required for heavy-tailed loss distributions, even pricing actuaries familiar with Choquet pricing are hesitant to use them. One reason may be the considerable choice of distortion operators [see Wang, 1996, for an overview]. Once a distortion operator has been selected, the task of calibrating in a least arbitrary (and most sensible) way must still be solved.

In the present work we try to overcome these obstacles by proposing both

1. a distortion operator that is optimal in the sense of minimizing relative entropy with respect to the actuarial probability measure [for justification and discussion, see e.g. Föllmer and Schweizer, 1991, or Gerber and Shiu, 1994];
2. a unique rule of calibration based on Lorenz order [see e.g. Yaari, 1987].

Using common notation, a set of insurable risky prospects  $\mathcal{X} \subset (\Omega, \mathcal{A}, \mathbb{P})$  and the pricing functional,

$$\begin{aligned} H : (\Omega, \mathcal{A}, \mathbb{P}) &\rightarrow \mathbb{R}^+ \\ X &\mapsto H(X), \end{aligned}$$

will be assumed. Expected values under the actuarial probability measure  $\mathbb{P}$  are referred to as actuarial expectation or actuarially fair premium, implying a pricing rule in accordance with the concept of a long-term pure risk premium that abstracts from insurance cycles or price shocks caused by temporary shortages of insurance capacity. The pricing concept presented in this work takes into account the pure loss dynamics underlying an insurance portfolio, risk aversion of market participants and the insurer's solvency. However, it neglects administrative expense, which is very much company-specific.

This paper is structured as follows. Section 2 presents the axiomatic framework that provides the basis for the theoretical results reviewed in section 3 and the practical application to follow in section 4. After applying the theory of quasi risk-neutral pricing to the Swiss Re exposure curves in section 4.1, premiums and Risk Adjusted Capital values are calculated for a real property insurance portfolio in section 4.2. Discussion and conclusions with an outlook to further research are given in the final sections 4.3 and 4.4.

## 2. AXIOMATIC FRAMEWORK AND ESTABLISHED RESULTS

For ease of reference, we briefly revisit the set of axioms to be satisfied by  $H$ . These axioms have been extensively discussed in literature [see e.g. Goovaerts et al, 1984].

**Axiom 1** [No Rip-off]. *For any risky prospect  $X \in \mathcal{X}$ , we require the pricing functional  $H$  to satisfy*

$$\mathbb{E}(X) \leq H(X) \leq \sup(X),$$

where

$$\sup(X) := \sup_{\omega \in \Omega} X(\omega)$$

If  $H(X) = \infty$ , a risk is assumed non-insurable.

**Axiom 2** [Monotonicity]. *The pricing functional  $H$  is required to preserve first-order stochastic order in that for any two risky prospects  $X_1$  and  $X_2$ , such that  $X_1 \leq_{st} X_2$ ,*

$$X_1 \leq_{st} X_2 \implies H(X_1) \leq H(X_2).$$

The notation “ $X_1 \leq_{st} X_2$ ” means that  $X_2$  exhibits first stochastic dominance over  $X_1$ , which in terms of distribution functions implies that

$$F_{X_1}(x) \geq F_{X_2}(x), \forall x \in \mathbb{R}^+.$$

It is noted that this monotonicity assumption is weaker than the concept of point-wise monotonicity as discussed in Wang’s papers, since it just relates to distribution functions rather than the underlying sample space. However, assuming point-wise monotonicity implies monotonicity with respect to first stochastic dominance so that this axiom is neither unrelated nor in contrast to Wang’s monotonicity axiom.

**Axiom 3** [Subadditivity]. *For any two risky prospects  $X_1$  and  $X_2$  it must hold true that*

$$H(X_1 + X_2) \leq H(X_1) + H(X_2).$$

**Axiom 4** [Comonotonic Additivity]. For any two comonotonic risky prospects  $X_1$  and  $X_2$ ,

$$H(X_1 + X_2) = H(X_1) + H(X_2).$$

Comonotonic risks exhibit perfect positive stochastic dependence<sup>1</sup>.

**Axiom 5** [Continuity]. For any increasing sequence of random prospects  $X_n(\omega) \nearrow X(\omega)$ ,  $\forall \omega \in \Omega$ ,

$$\lim_{n \rightarrow \infty} H(X_n) = H(X).$$

For later reference, we recall the representation of  $H$  as Choquet integral,

$$\begin{aligned} H(X) &= \int_{\mathbb{R}^+} x \pi'(F_X(x)) f_X(x) dx \\ &= \int_{\mathbb{R}^+} x d[\pi(F_X(x))]. \end{aligned} \tag{1}$$

where the distortion function  $\pi$  ( $\pi'$  denotes its derivative) mirrors the above axioms by the following properties,

( $\pi 1$ )  $\pi$  is increasing on  $(0, 1)$ ;

( $\pi 2$ )  $\pi$  is convex on  $(0, 1)$ ;

( $\pi 3$ )  $\pi[0] = 0$  and  $\pi[1] = 1$ .

Details showing how the above axioms 1 - 5 translate into these properties can for instance be found in Wang et al [1997]. An additional property advocated by Wang (2000),

( $\pi 4$ )  $\pi'[1] = \infty$ ,

deserves mentioning. It has some practical appeal because it counterbalances the decreasing likelihood of large losses, preventing expected loss levels in high layers from approaching zero. Also, as shown in Wang [2000], there is a distortion operator having property ( $\pi 4$ ) which reproduces the Black-Scholes formula for option pricing. However, property ( $\pi 4$ ) is not a logical consequence of the axiomatic framework presented above. Hence, it will not be considered in the further course of this paper.

---

<sup>1</sup>For details concerning the concept of comonotonicity, see e.g. Wang, Young, and Panjer [1997].

### 3. THE EXPONENTIAL DISTORTION OPERATOR

As stated in Section 1, this paper aims to overcome actuaries' reservation against Choquet pricing by proposing a particular distortion operator which is optimal in the sense of minimizing the relative entropy with respect to the actuarial probability measure. This criterion results in

$$\pi^*(q) = \frac{1 - e^{\lambda q}}{1 - e^\lambda}, \quad 0 \leq q \leq 1, \lambda \geq 0. \quad (2)$$

In order to assure convexity, as postulated by  $(\pi 2)$ , the parameter  $\lambda$  is non-negative. Details for obtaining (2), including a heuristic interpretation of  $\lambda$  as a risk aversion parameter, can be found in Niederau [2000, chapter 5 and Appendix B]. The second building block of the paper is a calibration of (2) based on Lorenz order.

**Theorem 1.** *For any two comonotonic and Lorenz ordered random prospects  $X_1$  and  $X_2$  (stated in the same currency), with  $\mathbb{E}(X_1) = \mathbb{E}(X_2) < \infty$  and  $X_1 \leq_L X_2$ , it holds true that*

$$\mathbb{E}(u(-X_1)) \geq \mathbb{E}(u(-X_2)) \iff H(X_1) \leq H(X_2), \quad (3)$$

for any increasing concave risk utility function  $u$ .

The “if” part in (3) is due to the stop-loss order preserving property of  $H$ , as shown by Hürlimann [1998]. For the “only if” part, the reader is referred to the proof in Niederau [2000, Appendix E].

**Remark 1.** *This Theorem can be restated equivalently with respect to stop-loss order, i.e. second-order stochastic dominance, since in the presence of equal expectations, Lorenz and stop-loss order are equivalent [see Shaked and Shantikumar, 1997].*

**Remark 2.** *Theorem 1 may appear restrictive due to the required equality in expectation. But even in presence of random prospects with unequal expected values (the usual case in reality), this apparent limitation can be overcome by using scaled random prospects. As can be verified [see e.g. Niederau, 2000, chapter 5], using the axiomatic framework revisited in Section 2, the functional  $H$  is positively homogeneous of first degree, i.e. for any  $X \in \mathcal{X}$  and  $\alpha \in \mathbb{R}^+$ , one has  $H(\alpha X) = \alpha H(X)$ . For scaling, let  $\beta \equiv 1/\mathbb{E}(X)$  to find that  $H(X) = \mathbb{E}(X) \cdot H(\beta X)$ . Accordingly, with  $\hat{X} := \beta X$ ,  $H(X)$  is determined uniquely by  $H(\hat{X})$ , provided the actuarial expectation of  $X$  exists. Hence, in view of Theorem 1, it is sufficient to concentrate on scaled random prospects for the remaining part of this work.*

**Remark 3.** *Just considering two random prospects in Theorem 1 focusses on partial Lorenz order. But this result can be extended by induction to any countable set of risky prospects which is complete under Lorenz order.*

**Remark 4.** *While Theorem 1 is a general result which applies to any distortion operator, a unique rule of calibration with respect to the exponential distortion operator of (2) is developed in Niederau [2000, chapter 5, Lemma 1]. What is shown here, with respect to the exponential distortion operator, is that (3) is equivalent to the existence of a bijective non-linear mapping,*

$$\begin{aligned} \lambda &: [0.5, 1) \rightarrow [0, \infty) \\ \xi &\quad \mapsto \lambda(\xi), \end{aligned} \tag{4}$$

which is determined by the implicit equation

$$(e^\lambda + \lambda e^\lambda + 1) / (\lambda (e^\lambda - 1)) = \xi, \tag{5}$$

with  $\xi$  denoting the point on  $[0, 1]$  at which the Lorenz function associated with some risk  $X \in \mathcal{X}$  assumes its mean value. For completeness [see Johnson and Kotz, 1995], note that the Lorenz function of some risk  $X$  is

$$L(p) = \int_0^p F_X^{-1}(u) du \Big/ \int_0^1 F_X^{-1}(u) du, \quad 0 \leq p \leq 1, \tag{6}$$

Letting  $Z := F_X \stackrel{d}{=} U(0, 1)$ , the very same ratio from (5) can be rewritten as

$$\mathbb{E}(Z e^{\lambda Z}) / \mathbb{E}(e^{\lambda Z}). \tag{7}$$

To conclude this review of theoretical results, the pricing rule obtained from (5) in essence relates to the Esscher transform as discussed by Bühlmann [1968], according to (7). Here, however, the Esscher transform is applied to the distribution function  $F_X$  rather than to the random prospect  $X$  itself. The difference between these two types of transform is discussed e.g. in Wang [2000]. Note that Theorem 1 holds true for all increasing concave risk utility functions. Therefore, the equivalence in (3) means that all risk-averse market actors order the two risks  $X_1$  and  $X_2$  as indicated by  $H$ . But since  $H$  has an expected value representation according to (1), a risk-averse market can be considered quasi risk-neutral with respect to the (tail-)weighted density  $f^*(x) := \pi'(F(x))f(x)$ . The qualifier “quasi” is used as

a sign of caution since in classical expected utility theory, risk neutrality refers to market actors who order risks uniquely with respect to their actuarial expectation, implying linear risk utility functions. However, as a consequence of axioms 1 and 2, reviewed in Section 2, market actors' risk preferences may be characterized by any increasing concave risk utility function, not only by linear ones, calling for extension of risk neutrality in the classical sense.

#### 4. A PRACTICAL PRICING APPLICATION

When it comes to practical application, there are many ad hoc methods to calibrate the distortion operator in (2). For instance, one way is to fix a layer premium for some basic layer (e.g. the first million Euro ) and calibrate  $\lambda$  to match a preset premium for that basic layer. This value of  $\lambda$  then determines the risk loading for any higher layer. However, this approach is judgemental in various respects. Even choosing a meaningful basic layer and its premium, assumes that the loss dynamics in the basic layer are indicative for the pricing of high-excess layers.

By way of contrast, a consistent rule of calibration is developed in Section 4.2 below, using Theorem 1 and its extension as alluded to in Remark 3. Rather than focussing on some layer for calibrating  $\lambda$ , this rule takes into account the shape of the distribution function over its whole domain by means of Lorenz order. In Section 4.2, this theory will be applied to a wind risk portfolio insured by a large European company. For modelling the loss severity of such a portfolio, a member of the Maxwell-Boltzmann, Boese-Einstein, Fermi-Dirac (MBEFD for short) class of distribution functions is used, as discussed by Bernegger [1997].

##### 4.1. Determining the Lorenz function of the Swiss Re exposure curve .

The Swiss Re exposure curve is nothing but the limited expected value function of a random prospect scaled to the unit interval. Its distribution function is given by

$$F_{b,g}(x) = 1 - \frac{1-b}{(g-1)b^{1-x} + (1-gb)}, \quad 0 \leq x < 1. \quad (8)$$

The parametrization proposed by Bernegger,

$$\begin{aligned} b &:= e^{3.1-0.15(1+c)c} \\ g &:= e^{(0.78+0.12c)c}, \end{aligned} \quad (9)$$

will also be used in the following. In particular, it reduces the representation in (8) to only one parameter which will be referred to as “ $c$  parameter”. Note that this

member of the MBEFD class of distribution functions is not defined for

$$\tilde{c} = -0.5 + \sqrt{0.25 + 3.1/0.15} \approx 4.07, \quad (10)$$

where  $F$  in (8) has another representation [see Bernegger, 1997, for details]. The distribution function (8) has the scaled argument  $x = y/MPL$ , where  $y$  relates to possible realizations of some underlying random prospect  $Y$  which is bound from above by its Maximum Possible Loss (MPL). Note that MPL does not necessarily mean total physical destruction; rather it stands for maximum damage if all risk protection and prevention measures fail. The calibration given in (9) reproduces certain pricing schemes used in industrial reinsurance markets. In particular, for  $c = 5$  one obtains the so-called ‘‘Lloyd’s scale’’, the scheme used by Lloyd’s. The following Lemma holds for a subset of values of  $c$  that define the range of loss severities that are considered insurable.

**Lemma 1.** *For real values  $0 < c \leq 10$ , distribution functions given by (8) are Lorenz ordered by their value of  $c$ .*

To remain focussed on our main subject, we only give a sketch of the

**Proof.** Formally, Lemma 1 states that

$$\forall i, \forall j, \forall x \ (0 < x < 1, \ (i, j) \in \mathcal{I}^2, \ 0 < c_i \leq c_j \leq 10) \implies L_{c_i}(x) \geq L_{c_j}(x), \quad (11)$$

where  $\mathcal{I}$  denotes some non-countable index set. To verify that the implication in (11) holds true, one needs to show the partial derivative  $\delta(x, c) := \frac{\partial}{\partial c} L_{g,b}(x)$  to be non-positive in the relevant domain  $[0, 1] \times (0, 10]$ . Thus, one seeks to determine the stationary point  $(x^*, c^*)$  where  $\delta$  takes its maximum. The proof is completed after verifying that  $\delta[x^*, c^*] \leq 0$ .

The Lorenz function  $L$  of  $F$  can then be determined from the representation in definition (6), using the fact that the inverse of  $F$  is given by

$$F^{-1}(q) = \begin{cases} 1 - \ln([(1-b)/(1-q) - (1-gb)]/[g-1])/\ln(b), & \text{if } q < 1 - \frac{1}{g} \\ 1, & \text{if } q \geq 1 - \frac{1}{g}. \end{cases}$$

with its expectation given by Bernegger [1997],

$$\frac{\ln(gb)(1-b)}{\ln(b)(1-gb)}. \quad (12)$$

To preserve space, the expressions for  $L$  and its mean value are not shown here.

**4.2. Calculation of prices and RAC in a quasi risk-neutral setting .** The data shown in table 1 below assume a selection of the industrial wind exposures in Belgium and the Netherlands. For the sake of focus, actuarial parameters in this case

TABLE 1 OVERVIEW OF EXPOSURE DATA AND PRICING RESULTS.

Loc	Ins Val	MPL	$c$	$\mathbb{E}(Z_i)$	$H(Z_i)$	RAC	RAC* (%)	RAC+ (%)
1	261.9	26.6	4.31	21.8	49.4	1.38	21.4	25.0
2	233.7	19.3	4.26	16.5	37.3	1.04	16.3	18.1
3	130.2	13.2	4.17	12.4	27.8	0.77	12.0	12.4
4	64.1	8.8	4.08	9.1	20.4	0.57	8.9	8.2
5	47.8	7.9	4.05	8.4	18.7	0.52	8.1	7.5
6	36.7	7.3	4.04	7.9	17.6	0.49	7.7	6.9
7	29.9	6.9	4.03	7.5	16.7	0.46	7.2	6.5
8	22.8	6.4	4.00	7.1	15.7	0.43	6.7	6.0
9	19.3	5.7	3.96	6.6	14.7	0.41	6.4	5.4
10	14.9	4.3	3.84	5.6	12.2	0.33	5.2	4.0
<i>Totals</i>	861.3	106.4	<i>n.a.</i>	102.4	230.0	6.40	100	100

study, i.e. those relating to the severity and frequency of storm losses, are assumed to be best-estimate and will not be discussed in any detail. The labels, “Loc” and “Ins Val” denote the location and the insured value, respectively. Throughout this section, all values relate to one year. Moreover, sums insured, MPL and RAC values are stated in million (mn.) of Euros, while (fair) premium values are stated in thousands of Euros. Due to the typical pathways of European cyclones, locations in Belgium and the Netherlands are usually exposed to the same storms, causing them to present comonotonic risks. The parameter  $c$  is increasing in the MPL, reflecting that the probability of total loss of a building is decreasing with its MPL, in accordance with the distribution function (8). For all locations the value of  $c$  differs from the critical value of 4.07, as in (10), indicating that the distribution function in (8) is well defined. Therefore the data satisfy all prerequisites of Theorem 1 and Remark 3, since (besides comonotonicity) these ten locations are completely Lorenz ordered by their value of  $c$ , according to Lemma 1. A full value cover is assumed for the sake of simplicity, meaning that neither the insured nor the insurer

impose any loss limits other than the location-specific MPL<sup>2</sup>. A common expected loss frequency of 0.35 percent per location and per annum was assumed<sup>3</sup>. This expected loss frequency is low enough to justify the use of a diatomic distribution with parameters  $p_i = 0.035$  and  $n_i = 1, \forall i = 1, \dots, 10$ , respectively<sup>4</sup>. The diatomic representation of the loss frequency implies that with probability  $p_i$  one loss occurs at location  $i$  and accordingly, with probability  $1 - p_i$  no loss occurs. Thus, treating each location as a single risk unit  $X_i$ , total portfolio loss can be written as

$$Z = \sum_{i=1}^{10} Z_i = \sum_{i=1}^{10} \mathbb{1}_{\{N=1\}} X_i, \quad (13)$$

where

$$H(Z) = \sum_{i=1}^{10} [p_i H(X_i) + (1 - p_i) H(0)] = \sum_{i=1}^{10} \left[ p_i H(\hat{X}_i) \mathbb{E}(X_i) \right]. \quad (14)$$

In accordance with the notation introduced before,  $\hat{X}_i$  refers to the scaled loss variable  $X_i/\mathbb{E}(X_i)$ . Here, the pricing functional  $H$  takes into account that no loss occurs with probability  $1 - p_i$ , which does not give rise to any premium (in keeping with Axiom 1). The total expected loss per location is obtained by Wald's identity, i.e.  $\mathbb{E}(Z_i) = p_i \mathbb{E}(X_i)$ . Recalling (14),  $H(Z_i)$  is nothing but  $p_i H(X_i)$ , which justifies

$$H(Z_i)/\mathbb{E}(Z_i) = H(X_i)/\mathbb{E}(X_i).$$

The latter ratio can thus be interpreted as the loading factor charged for location  $i$  [see table 3 in the Appendix for a set of loading factors as a function of  $c$ ].

The determination of the (one-year) Risk Adjusted Capital (RAC), or more precisely, the return on Risk Adjusted Capital (RoRAC), is an important issue to insurers, since (in conjunction with the insurer's  $\beta$  from the CAPM) it helps investors to position an insurer relative to their efficient portfolio frontier [see for details e.g.

---

<sup>2</sup>This is clearly an abstraction from reality since insurers would usually sublimit their exposures to natural perils in order to counteract loss accumulation.

<sup>3</sup>It should be noted that the assumed expected frequency of 1 in 30 years (roughly) does not relate to regular storms but to cyclones with wind speeds in the order of magnitude of about 150 km/h or more that have the potential to cause serious damage even to industrial buildings, such as considered in this case study.

<sup>4</sup>The only reason for having chosen a diatomic distribution with preference to a Poisson distribution is to ease the computation of  $H$ . For more details on the calculation of  $H$  for compound Poisson distributions the reader is referred to Niederau [2000, Appendix C].

Zweifel and Auckenthaler, 2007]. Now recall that under quasi risk-neutrality, the location-specific surcharges  $H(Z_i) - \mathbb{E}(Z_i)$  can be interpreted as certainty-equivalent net cashflows. This permits to uniquely determine the respective present value of RAC using

$$RAC_i = [H(Z_i) - \mathbb{E}(Z_i)] / r_f, \quad (15)$$

where  $r_f$  denotes the one-year risk-free hurdle rate. The according RAC values are shown in table 1 for an assumed hurdle rate of 2 percent per annum. Total aggregated RAC, being defined as

$$RAC_{tot} = \sum_{i=1}^{10} RAC_i,$$

amounts to about 6 percent of total MPL, or Euro 6.4 mn. Table 2 shows that this value is close to the 99.6% quantile of the aggregate loss distribution of the portfolio risk as represented by (13). This implies that only once in 250 years will

TABLE 2 SELECTED PERCENTILES OF THE AGGREGATE PORTFOLIO DISTRIBUTION.

loss amount	percentile
0	96.50
0.71	97.00
1.08	97.50
1.47	98.00
1.98	98.50
3.32	99.00
5.89	99.50
6.45	99.60
8.54	99.75
12.11	99.90

the aggregate loss burden exceed  $RAC_{tot}$ , forcing the insurer to raise capital beyond this value.

It may be worthwhile to emphasize the sensitivity of these estimates to the cutoff point chosen. At the 95% quantile, no RAC would have been necessary, while at 99% quantile (a preferred choice in practical applications), RAC would amount to some Euro 3.2 mn., just about one-half of the calculated value [see table 2]. More generally,

quantile-based rules for the determination of RAC may expose an insurer to a great potential for error, particularly in the presence of highly skewed loss distributions. However, even when using the exact RAC calculation rule, insurers may still be concerned about parameter uncertainty, a topic which is not addressed in this paper but merits mentioning [for details on the treatment of parameter uncertainty see e.g. Wang, 2003]. In table 1, the allocation as derived from the quasi risk-neutral pricing (indicated by  $RAC^*$ , in percent of  $RAC_{tot}$ ), is juxtaposed to a typical practitioner's rule-of-thumb allocation  $RAC^+$ , keeping total RAC constant at Euro 6.4 mn. The rule of thumb allocates RAC according to the location's relative MPL, such that

$$\frac{RAC_i^+}{RAC_{tot}} = \frac{MPL_i}{\sum_{i=1}^{10} MPL_i}.$$

As can be seen from the last column of table 1, this rule biases RAC allocation in favor of the locations with high MPL values. By way of comparison, the allocation based on quasi risk-neutral pricing results in a more balanced RAC allocation. Since RAC is often used not only as a measure of risk tolerance but also for performance measurement, choosing an appropriate allocation rule is of considerable importance. The rule based on quasi risk-neutral pricing can be argued to be preferable because it takes full account of the probabilistic features of the risks involved rather than just focussing on the maximum (foreseeable) exposure to loss, a mere reflection of risk aversion. In the present context it recognizes the fact that losses in property insurance do not necessarily occur at locations with maximum MPL but also hit locations of medium to small size.

**4.3. Discussion.** While providing interesting insights, this case study of course cannot claim general validity. Usually, insurance portfolios are neither completely Lorenz ordered nor are they comonotonic. However, Theorem 1 can be generalized beyond comonotonicity and Lorenz order in the following sense. Denote with  $X_1$  and  $X_2$  two single risks (e.g. two locations as in the practical application). Then Axiom 3 implies

$$H(X_1 + X_2) \leq H(X_1) + H(X_2). \quad (16)$$

In turn, provided  $X_1$  and  $X_2$  have finite expectation,  $H(X_1)$  and  $H(X_2)$  are uniquely determined by  $H(\hat{X}_1)$  and  $H(\hat{X}_2)$ . The latter quantities can be uniquely calculated, following the logic of Theorem 1 and the calibration motivated in Remark 4, since  $X_1$  and  $X_2$  are both partially Lorenz ordered and comonotonic with respect to themselves, respectively.

These considerations can be used to calculate at least an upper bound RAC for *any* insurance portfolio. If  $PF^{(I)} = (Y_i)_{i=1,\dots,n}$  denotes some insurance portfolio,  $Y_i$  single risks for all  $i$ , and  $RAC(PF^{(I)})$  the one-year RAC for this portfolio, then the theory predicts

$$RAC(PF^{(I)}) \leq \sum_{i=1}^n [H(Y_i) - \mathbb{E}(Y_i)] / r_f \quad (17)$$

This inequality is of interest for at least three reasons. First, from a risk management point of view, it provides a value of the maximum RAC assuming a complete lack of diversification effects in the portfolio under consideration. Second, the ratio

$$\frac{r_f RAC(PF^{(I)})}{\sum_{i=1}^n [H(Y_i) - \mathbb{E}(Y_i)]} \leq 1 \quad (18)$$

may serve as an operational measure of diversification effects. Indeed, (18) can be interpreted as the benefit of diversification to potential purchasers of insurance. The higher this benefit the closer to zero the ratio on the left-hand side of (18) is. Third, from a shareholder value point of view, excessive RAC, indicated by a violation of inequality (18), is a cause of concern. Too much costly capital would be tied up by underwriting and as a consequence, return to capital is both understated and lower than necessary, to the detriment of shareholders. However, this means that an insurance company runs the risk of not being on investors' efficient frontier in terms of expected returns and volatility of returns, unless this shortfall can be made up by a success in capital investments that outperforms the other investors in the capital market [see e.g. Zweifel and Auckenthaler, 2007].

**4.4. Conclusions and outlook.** The objective of this contribution is to derive a quasi risk-neutral pricing rule for insurance that amounts to an expected value, defined over a modified probability measure. While assuring minimum relative entropy with respect to the actuarial probability measure, this modification reflects risk aversion of market actors in the insurance industry. To the extent that insurance risks are comonotonic and exhibit Lorenz order, loading factors for the insurer's Risk Adjusted Capital can be uniquely determined. They call for a higher surcharge for those (scaled) risks dominating under Lorenz order. Moreover, a maximum price can be derived for any portfolio by abstracting from diversification effects, i.e. by pricing every single risk unit. Quasi risk-neutral pricing is applied to the property risk of a set of industrial plants having exposure to storm loss. Location-specific

RAC values are calculated and compared to a typical practitioner’s rule that allocates RAC according to the location’s maximum possible loss (MPL) in proportion to the portfolio’s total MPL. Since quasi risk-neutral pricing is based not only on maximum exposure to loss but also takes into account the shape of the loss distribution, it avoids underreserving for risks that have considerable loss potential in the small to medium range which fails to be reflected in their limited MPL values.

Several open questions remain for future research. First, while proving sufficient, the conditions imposed in Theorem 1 to obtain a quasi risk-neutral pricing rule may not be necessary. Second, the maximum price derived for any portfolio may be refined to yield more accurate upper bounds (or even exact values) by including the effects of risk diversification. In the same vein, insurers’ capital investments should be taken into account because they provide additional hedging opportunities. These considerations might result in a more general rule for RAC allocation, with RAC determined by a coherent risk measure in the sense of Artzner et al. [1999].

#### APPENDIX

As show in table 3, the sensitivity of the loading factors is highest for values of  $c$  up to about 3.5. For higher values of  $c$ , such as considered in Section 4.2, they still increase but at a decreasing rate. These values may appear counter-intuitive to insurance practitioners who would usually consider a risk characterized by  $c = 1$  as being more risky than one with  $c = 5$  (e.g.) because the former exhibits more exposure to MPL and hence seems to require more risk capital. This apparent paradox can be explained by recalling the separation of scale and risk, mentioned in Remark 2. According to the parametrization in (5), when  $c$  approaches zero this means approaching the deterministic case, i.e. MPL being realized with certainty<sup>5</sup>. While depending on the size of the MPL, such an exposure to loss can call for a great deal of capital. However, in keeping with axiom 1, this capital will be completely provided by the premium charged, which coincides with the MPL and therefore with the actuarially fair premium. Hence, in the limit, when  $c$  approaches zero, there is no need or even justification to charge a premium beyond the actuarially fair value. Conversely, a risk with  $c = 5$  exhibits quite a skewed loss distribution and the actuarially fair premium is but 1.22 percent of the MPL, regardless of its

---

<sup>5</sup>Amunition plants or any industrial plant exposed to vapor cloud explosion may serve as an example.

TABLE 3 LOADING FACTORS AND RELATED PARAMETERS AS A FUNCTION OF  $c$ .

c-value	xi	lambda	loading factor
1.0	0.5784	0.955	1.21
2.0	0.6199	1.492	1.45
2.5	0.6335	1.676	1.62
3.0	0.6644	2.115	1.97
3.5	0.6771	2.307	2.12
4.0	0.6856	2.439	2.22
4.5	0.6905	2.517	2.289
5.0	0.6932	2.560	2.331
5.5	0.6946	2.583	2.36
6.0	0.6953	2.595	2.37
6.5	0.6957	2.601	2.39
7.0	0.6960	2.606	2.40

value<sup>6</sup>. In case of such a risk, the insurer needs to raise extra risk capital in order to fund a potential MPL. The associated opportunity cost of risk capital is reflected in the loading factor of 2.33, as shown in table 3. A set of comparative loading factors for the generalized Pareto distribution, discussed extensively in Embrechts et al. [1997], can be found in Niederau [2000, Appendix C].

#### ACKNOWLEDGEMENTS

We are very thankful for the enriching comments and hints on behalf of Prof. Hans Bühlmann, Prof. Paul Embrechts, and Prof. Freddy Delbaen while this contribution took shape.

#### REFERENCES

- [1] Artzner, P., Delbaen, F., Eber, J.M., Heath, D., 1999, Coherent Measures of Risk, in: *Mathematical Finance*, 9, (3), 203-228.
- [2] Bernegger, S., 1997, The Swiss Re Exposure Curves and MBBEFD Distribution Class, in: *Astin Bulletin*, 27, (1), 99-111.

---

<sup>6</sup>This value can be verified using formula (12).

- [3] Bühlmann, H., 1980, An Economic Premium Principle, in: *Astin Bulletin*, 11, 52-60.
- [4] Delbaen, F, Haezendonck, J., 1989, A Martingale Approach to Premium Calculation Principles in an Arbitrage-Free Market, in: *Insurance: Mathematics and Economics*, 8 (4), 269-277.
- [5] Denneberg, D., 1990, Premium Calculation: Why Standard Deviation Should Be Replaced by Absolute Deviation, *Astin Bulletin*, 20, (2), 181-190.
- [6] Embrechts, P., Klüppelberg, C., Mikosch, T., 1997, *Modelling Extremal Events for Insurance and Finance*, Heidelberg: Springer.
- [7] Föllmer, H., Schweizer, M., 1991, Hedging of Contingent Claims Under Incomplete Information, in: *Applied Stochastic Analysis* (Davis, M. H. and Elliott, R.J., eds.), 389-414, Gordon and Breach.
- [8] Gerber, H. U., Shiu, E. S., 1994, Option Pricing by Esscher Transforms (with discussion), in: *Transactions of the Society of Actuaries* 46, 99-191.
- [9] Goovaerts, M. J., De Vijlder, F., Haezendonck, J., 1984, *Insurance Premiums: Theory and Applications*, Amsterdam: North-Holland Publishing Co.
- [10] Hürlimann, W., 1998, On Stop-Loss Order and the Distortion Pricing Principle, in: *Astin Bulletin* 28, (1), 119-134.
- [11] Johnson, N.L., Kotz, S., Balakrishnan, N., 1995, *Continuous Univariate Distributions*, New York: John Wiley.
- [12] Niederau, H., 2000, *Pricing Risk in Incomplete Markets: An Application to Industrial Reinsurance*, Doctoral Thesis at the Socioeconomic Institute at the University of Zurich.
- [13] Shaked, M., Shantikumar, J.G., 1997, *Stochastic Orders and their Applications*, Boston: Academic Press.
- [14] de Waegenaere, A., Kast, R., Lapiéd, A., 2003, Choquet Pricing and Equilibrium, in: *Insurance: Mathematics and Economics*, 32, (3), 359-370.
- [15] Wang, S., 2003, Cat Bond Pricing using Probability Transforms, in: *Geneva Papers* 278, 19-29.

- [16] Wang, S., 2000, A Class of Distortion Operators for Pricing Financial and Insurance Risks, in: *Journal of Risk and Insurance*, 67, (1), 15-36.
- [17] Wang, S. S., Young, V. R., Panjer, H. H., 1997, Axiomatic Characterization of Insurance Prices, in: *Mathematics and Economics*, 21, (2), 173-183.
- [18] Wang, S., 1996, Premium Calculation by Transforming the Layer Premium Density, in: *Astin Bulletin*, 26, (1), 71-92.
- [19] Wang, S., 1995, Insurance Pricing and Increased Limits Ratemaking by Proportional Hazard Transforms, in: *Insurance: Mathematics and Economics*, 17, 43-54.
- [20] Yaari, M. E., 1987, The Dual Theory of Choice Under Risk, in: *Econometrica*, 55, (1), 95-115.
- [21] Zweifel, P., Auckenthaler, Ch., 2007, On the Feasibility of Insurer's Investment Policies, resubmitted to: *Journal of Risk and Insurance*.

*Harry Niederau*

Dr. Niederau Consulting&Research  
hniederau@niederau-research.com

*Peter Zweifel*

Socioeconomic Institute  
University of Zurich  
pzweifel@soi.unizh.ch